Topology II

Fourth Year

Second Semester

Definition : Open & Closed Functions

Let $f : (X, \tau) \to (Y, \tau')$ be a function.

(1) The function f is called **open** if the direct image for any open set in X is open set in Y. i.e.,

$$f : (X, \tau) \to (Y, \tau') \text{ is open function } \Leftrightarrow \forall\, U \in \tau \Rightarrow f(U) \in \tau'$$

$$f : (X, \tau) \to (Y, \tau') \text{ is not open function } \Leftrightarrow \exists\, U \in \tau \wedge f(U) \notin \tau'$$

(2) The function f is called **closed** if the direct image for any closed set in X is closed set in Y. i.e.,

$$f : (X, \tau) \to (Y, \tau') \text{ is closed function } \Leftrightarrow \forall\, F^c \in \tau \Rightarrow (f(F))^c \in \tau'$$

$$f : (X, \tau) \to (Y, \tau') \text{ is not closed function } \Leftrightarrow \exists\, F^c \in \tau \wedge (f(F))^c \notin \tau'$$

Example (1) : (T T T) means the function f is continuous, open , closed.

Define the identity function $f : (X, \tau) \to (X, \tau)$; $f(x) = x \quad \forall\, x \in X$.

f is continuous since $\forall\ V \in \tau$ in rang X $\Rightarrow$ $f^{-1}(V) = V \in \tau$ in domain X.

f is open since $\forall\ U \in \tau$ open in domain X $\Rightarrow$ $f(U) = U$ is open in rang X.

f is closed since $\forall$ F closed in domain X $\Rightarrow$ $f(F) = F$ is closed in rang X.

Example (2) : (T T F) means the function f is continuous, open , not closed.

Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{a\}\}$

Define the constant function $f : (X, \tau) \to (Y, \tau')$; $f(1) = f(2) = f(3) = a$

f is continuous since it is constant.

f is open since : $X \in \tau \Rightarrow f(X) = \{a\} \in \tau'$, $\phi \in \tau \Rightarrow f(\phi) = \phi \in \tau'$ and $\{1\} \in \tau \Rightarrow f(\{1\}) = \{a\} \in \tau'$ (i.e., $\forall\ U \in \tau \Rightarrow f(U) \in \tau'$).

f is not closed since $\exists$ closed set $\{2, 3\} \in \mathcal{F}$ (since $\{2, 3\}^c = \{1\} \in \tau$)

But, $f(\{2, 3\}) = \{a\} \notin \mathcal{F}'$ since $\{a\}^c = \{b, c\} \notin \tau'$

Example (3) : (T F T) means the function f is continuous, not open , closed.

Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{a, b\}\}$

Define the constant function $f : (X, \tau) \to (Y, \tau')$; $f(1) = f(2) = f(3) = c$

f is continuous since it is constant.

f is not open since $\exists$ open set $\{1\} \in \tau$,but $f(\{1\}) = \{c\} \notin \tau'$.

f is closed since : The family of closed sets in X is $\mathcal{F} = \{X, \phi, \{2, 3\}\}$ and the family of closed set in Y is $\mathcal{F}' = \{Y, \phi, \{c\}\}$, then

$f(X) = \{c\}$, $f(\phi) = \phi$, $f(\{2, 3\}) = \{c\}$ (i.e., $\forall\ F \in \mathcal{F} \Rightarrow f(F) \in \mathcal{F}'$).

Example (4) : (F T T) means the function f is not continuous, open , closed.

Define the function $f : (\mathbb{R}, I) \to (\mathbb{R}, \tau_u)$; $f(x) = x \quad \forall x \in \mathbb{R}$

f is not continuous since $\exists$ (0, 1) open in $(\mathbb{R}, \tau_u)$, but $f^{-1}((0, 1)) = (0, 1)$ not open in $(\mathbb{R}, I)$.

f is open and closed since the only open and closed sets in $(\mathbb{R}, I)$ are $\mathbb{R}$, ϕ and $f(\mathbb{R}) = \mathbb{R}$ and $f(\phi) = \phi$ (i.e., the direct image of open (rep., closed) set is open (rep., closed) set)

Example (5) : (T F F) means the function f is continuous, not open , not closed.

Define the function $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, I)$; $f(x) = x \quad \forall x \in \mathbb{R}$

f is continuous since the rang is $(\mathbb{R}, I)$

f is not open since $\exists$ open set (0, 1) in $(\mathbb{R}, \tau_u)$, but $f((0, 1)) = (0, 1)$ is not open in $(\mathbb{R}, I)$.

f is not closed since $\exists$ closed set {0} in $(\mathbb{R}, \tau_u)$, but $f(\{0\}) = \{0\}$ is not closed in $(\mathbb{R}, I)$.

Remark : The open function is closed and the closed function is open if the function is bijective (injective and surjective). i.e.,

$$f \text{ is bijective function} \Rightarrow (f \text{ open} \Leftrightarrow f \text{ closed})$$

$$f \text{ is bijective function} \Rightarrow (f \text{ not open} \Leftrightarrow f \text{ not closed})$$

This means if we wanted to get a function is open not closed or closed not open must be define a function not bijection (not injective or not surjective) since if we define a bijective function then it's either open and closed or not open and not closed.

Example : Define the function $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, I)$; $f(x) = x \quad \forall x \in \mathbb{R}$

Notes that f is continuous since the rang is $(\mathbb{R}, I)$

Notes that f is bijective , then f^{-1} is exists function and

$f^{-1} : (\mathbb{R}, I) \to (\mathbb{R}, \tau_u)$, but this function not continuous since

$\exists$ open set (0, 1) in $(\mathbb{R}, \tau_u)$, but $(f^{-1})^{-1}((0, 1)) = (0, 1)$ is not open in $(\mathbb{R}, I)$.

Definition : Homeomorphism Functions

Let $f : (X, \tau) \to (Y, \tau')$ be a function. The function f is called **homeomorphism** if its injective, surjective, continuous and f^{-1} continuous. i.e.,

$f : (X, \tau) \to (Y, \tau')$ is homeomorphism $\Leftrightarrow$ f 1-1, onto, continuous and f^{-1} continuous

$f : (X, \tau) \to (Y, \tau')$ is not homeomorphism $\Leftrightarrow$

$$f \text{ not 1-1} \vee f \text{ not onto} \vee f \text{ not continuous} \vee f^{-1} \text{ not continuous}$$

Remark : Clear that every homeomorphism function is continuous, but the converse is not true for example :

$$f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, I) \ ; \ f(x) = x \quad \forall \ x \in X$$

The function f is1-1, onto, continuous, but f^{-1} is not continuous. Therefore f is not homeomorphism.

Remark : If $f : (X, \tau) \to (Y, \tau')$ is homeomorphism function, this means : $(f^{-1})^{-1}(U) \in \tau' \quad \forall \ U \in \tau$ (def of continuity), but $(f^{-1})^{-1}(U) = U$ (since f bijective), so we can said

$$f^{-1} \text{ is continuous } \Leftrightarrow \ f(U) \in \tau' \quad \forall \ U \in \tau$$

but this is the definition of open function, so if f^{-1} is continuous this means f is open and vise versa with property that f is bijective. i.e.,

$$f^{-1} \text{ is continuous } \Leftrightarrow \ f \text{ is open}$$

if f is bijective (by previous remark, p. 47, f is open $\Leftrightarrow$ f is closed), so that

$$f^{-1} \text{ is continuous } \Leftrightarrow \ f \text{ is open } \Leftrightarrow \ f \text{ is closed}$$

i.e., the three concepts are equivalent and we can replace the definition of homeomorphism as follow :

$$f \text{ is homeomorphism} \Leftrightarrow f \text{ is 1-1, onto, continuous and open.}$$

$$f \text{ is homeomorphism} \Leftrightarrow f \text{ is 1-1, onto, continuous and closed.}$$

such that we replace the statement f^{-1} is continuous in definition of homeomorphism by either f open function or f closed function.

Remarks :

[1] If f is homeomorphism function, then f^{-1} is also homeomorphism function.
since f is 1-1 and onto, then f^{-1} is 1-1 and onto
since f is homeomorphism, then f^{-1} is continuous, also, $f = (f^{-1})^{-1}$ is continuous
Therefore, f^{-1} is homeomorphism function.

[2] If $f : (X, \tau) \to (Y, \tau')$ and $g : (Y, \tau') \to (Z, \tau'')$ are both homeomorphism functions, then the composition gof : $(X, \tau) \to (Z, \tau'')$ is homeomorphism.
since f and g are 1-1 and onto, then gof is 1-1 and onto
since f and g are continuous, then gof is continuous (by previous theorem)
since f and g are homeomorphism, then f^{-1} and g^{-1} are continuous also $f^{-1}og^{-1}$ is continuous (by previous theorem)
but, $f^{-1}og^{-1} = (gof)^{-1}$ is continuous.
Therefore, gof is homeomorphism function.

Definition : Homeomorphic Topologies

We called two topological spaces (X, τ) and (Y, τ') are **homeomorphic** if there exists a homeomorphism function from (X, τ) to (Y, τ') and denoted by $(X, \tau) \cong (Y, \tau')$ or $(Y, \tau') \cong (X, \tau)$. i.e.,

$$(X, \tau) \cong (Y, \tau') \Leftrightarrow \exists \text{ homeomorphism function } f : (X, \tau) \to (Y, \tau')$$

Theorem : The relation $\cong$ is an equivalent relation on the family of topologic spaces.

Proof : We must prove the relation $\cong$ is reflexive, symmetric and transitive.

(1) To prove $\cong$ is reflexive. i.e., $(X, \tau) \cong (X, \tau)$??

Define the identity function $f : (X, \tau) \to (X, \tau)$; $f(x) = x \quad \forall x \in X$

Clear that f is 1-1, onto, continuous and $f = f^{-1}$ so that f^{-1} is continuous

Therefore, $\cong$ is reflexive.

(2) To prove $\cong$ is symmetric. i.e., if $(X, \tau) \cong (Y, \tau') \Rightarrow (Y, \tau') \cong (X, \tau)$??

$\because (X, \tau) \cong (Y, \tau') \Rightarrow \exists$ homo. funct. $f : (X, \tau) \to (Y, \tau')$

by remark [1] above, we have f^{-1} is homo. funct. and

$f^{-1} : (Y, \tau') \to (X, \tau) \Rightarrow (Y, \tau') \cong (X, \tau)$

Therefore, $\cong$ is symmetric.

(3) To prove $\cong$ is transitive. i.e., if $(X, \tau) \cong (Y, \tau') \cong (Z, \tau'') \Rightarrow (X, \tau) \cong (Z, \tau'')$??

$\because (X, \tau) \cong (Y, \tau') \Rightarrow \exists$ homo. funct. $f : (X, \tau) \to (Y, \tau')$ and

$\because (Y, \tau') \cong (Z, \tau'') \Rightarrow \exists$ homo. funct. $g : (Y, \tau') \to (Z, \tau'')$ and

by remark [2] above, we have gof is homo. funct. and

$gof : (X, \tau) \to (Z, \tau'') \Rightarrow (X, \tau) \cong (Z, \tau'')$

Therefore, $\cong$ is transitive.

Now, the following theorem for continuous functions is needed with its proof beform introducing an additional property of homemorphic topological spaces:

theorem :

Theorem : Let $f : (X, \tau) \to (Y, \tau')$ be a function. Then f is continuous iff satisfy one of the following properties :

(1) $f^{-1}(F) \in \mathcal{F} \quad \forall \quad F \in \mathcal{F}'$; $\mathcal{F}$ family of closed sets in X and $\mathcal{F}'$ family of closed sets

(2) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$; $B \subseteq Y$.

(3) $f^{-1}(B^{o}) \subseteq (f^{-1}(B))^{o}$; $B \subseteq Y$.

Proof :

(1) To prove, f is continuous $\Leftrightarrow$ $X - f^{-1}(F) \in \tau \quad \forall \quad Y - F \in \tau'$

($\Rightarrow$) Suppose that f is cont. , to prove $X - f^{-1}(F) \in \tau \quad \forall \quad Y - F \in \tau'$

Let F closed set in Y $\Rightarrow$ Y – F open set in Y (def. of closed set)

$\Rightarrow f^{-1}(Y - F) \in \tau$ (since f is continuous)

But, $f^{-1}(Y - F) = f^{-1}(Y) - f^{-1}(F) = X - f^{-1}(F)$ (since $f^{-1}(Y) = X$)

$\because \quad f^{-1}(Y - F) \in \tau \quad \Rightarrow \quad X - f^{-1}(F) \in \tau$

($\Leftarrow$) Suppose that $X - f^{-1}(F) \in \tau \quad \forall \quad Y - F \in \tau'$, to prove f is continuous

Let V open set in Y i.e., $V \in \tau'$

$\therefore$ Y – V closed set in Y since V open

$\Rightarrow f^{-1}(Y - V)$ closed set in X (by hypothesis)

$\Rightarrow X - f^{-1}(Y - V) \in \tau$

But, $X - f^{-1}(Y - V) = X - [f^{-1}(Y) - f^{-1}(V)] = X - [X - f^{-1}(V)] = f^{-1}(V)$

$\Rightarrow f^{-1}(V) \in \tau$

$\therefore$ f is continuous.

(2) To prove, f is continuous $\Leftrightarrow$ $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$; $B \subseteq Y$.

($\Rightarrow$) Suppose that f is continuous , to prove $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$; $B \subseteq Y$.

Let $B \subseteq Y \Rightarrow B \subseteq \overline{B} \Rightarrow f^{-1}(B) \subseteq f^{-1}(\overline{B})$

$\because \overline{B}$ closed set in Y by (1) $f^{-1}(\overline{B})$ closed set in X

$\Rightarrow \bigcap\{F \subseteq X : F^c \in \tau \wedge f^{-1}(B) \subseteq F\} \subseteq f^{-1}(\overline{B})$

since $\overline{f^{-1}(B)}$ is intersection of all closed sets that contain $f^{-1}(B)$ and $f^{-1}(\overline{B})$ is one of the closed set that contain $f^{-1}(B)$, then

$\Rightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ (since $\bigcap\{F : F^c \in \tau \wedge f^{-1}(B) \subseteq F\} = \overline{f^{-1}(B)}$)

($\Leftarrow$) Suppose that $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$; $B \subseteq Y$. , to prove f is continuous.

We will use part (1) in this theorem

Let F closed set in Y, to prove $f^{-1}(F)$ closed in X i.e., $\overline{f^{-1}(F)} = f^{-1}(F)$.

$f^{-1}(F) \subseteq \overline{f^{-1}(F)}$ (since $A \subseteq \overline{A}$) ----------(1)

$\because$ F closed $\Rightarrow F = \overline{F} \Rightarrow f^{-1}(F) = f^{-1}(\overline{F})$

By hypnoses, $\overline{f^{-1}(F)} \subseteq f^{-1}(\overline{F}) = f^{-1}(F)$

$\Rightarrow \overline{f^{-1}(F)} \subseteq f^{-1}(F)$ ----------(2)

From (1) and (2) we have $\overline{f^{-1}(F)} = f^{-1}(F)$

$\therefore f^{-1}(F)$ closed in X. $\therefore$ f is continuous.

(3) To prove, f is continuous $\Leftrightarrow f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$; $B \subseteq Y$.

($\Rightarrow$) Suppose that f is continuous , to prove $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$; $B \subseteq Y$

Let $B \subseteq Y \Rightarrow B^\circ \subseteq B \Rightarrow f^{-1}(B^\circ) \subseteq f^{-1}(B)$

$\because$ B° is open in Y $\Rightarrow f^{-1}(B^\circ)$ is open in X (since f is continuous)

$f^{-1}(B^\circ) \subseteq \bigcup\{O \subseteq X ; O \in \tau , O \subseteq f^{-1}(B)\}$

since $(f^{-1}(B))^\circ$ is union of all open sets that contain in $f^{-1}(B)$ and $f^{-1}(B^\circ)$ is one of the open set that contain in $f^{-1}(B)$, then

$\Rightarrow f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$ (since $\bigcup\{O \subseteq X ; O \in \tau , O \subseteq f^{-1}(B)\} = (f^{-1}(B))^\circ$)

($\Leftarrow$) Suppose that $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$; $B \subseteq Y$, to prove f is continuous.

Let V open set in Y i.e., $V \in \tau'$ to prove $f^{-1}(V)$ open in X i.e., $f^{-1}(V) = (f^{-1}(V))^\circ$.

$\because$ $V^\circ \subseteq V \Rightarrow ((f^{-1}(V))^\circ \subseteq f^{-1}(V)$ ----------(1)

$\because$ V open $\Rightarrow V = V^\circ \Rightarrow f^{-1}(V) = f^{-1}(V^\circ)$

By hypnoses, $f^{-1}(V) = f^{-1}(V^\circ) \subseteq (f^{-1}(V))^\circ$

$\Rightarrow f^{-1}(V) \subseteq (f^{-1}(V))^\circ$ ----------(2)

From (1) and (2) we have $f^{-1}(V) = (f^{-1}(V))^\circ$

$\therefore$ $f^{-1}(V)$ open in X

$\therefore$ f is continuous.

Theorem :

(1) The bijective function $f : (X, \tau) \to (Y, \tau')$ is homeomorphism iff $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$; $B \subseteq Y$.

(2) The bijective function $f : (X, \tau) \to (Y, \tau')$ is homeomorphism iff $f^{-1}(B^\circ) = (f^{-1}(B))^\circ$; $B \subseteq Y$.

Proof :

(1) ($\Leftarrow$) Suppose that $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$, to prove f Home.,

$\because$ f is bij., we must prove f is cont. and f^{-1} is cont.

$\because$ $\overline{f^{-1}(B)} = f^{-1}(\overline{B}) \Rightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) \Rightarrow$ f is cont.

(by theory f is cont $\Leftrightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$; $B \subseteq Y$)

and $\because$ $\overline{f^{-1}(B)} = f^{-1}(\overline{B}) \Rightarrow f^{-1}(\overline{B}) \subseteq \overline{f^{-1}(B)} \Rightarrow f^{-1}$ is cont.

(by theory f is cont $\Leftrightarrow f(\overline{B}) \subseteq \overline{f(B)}$; $B \subseteq X$ and f replace by f^{-1}) $\therefore$ f is Home.

($\Rightarrow$) Suppose that f is Home., to prove $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$

$\because$ f is home. $\Rightarrow$ f is cont. and f^{-1} is cont.

$\Rightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ and $f^{-1}(\overline{B}) \subseteq \overline{f^{-1}(B)} \Rightarrow \overline{f^{-1}(B)} = f^{-1}(\overline{B})$

(2) ($\Leftarrow$) Suppose that $f^{-1}(B^o) = (f^{-1}(B))^o$, to prove f Home.,

$\because$ f is bij., we must prove f is cont. and f^{-1} is cont.

$\because f^{-1}(B^o) = (f^{-1}(B))^o \Rightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o \Rightarrow$ f is cont.

(by theory f is cont $\Leftrightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o$; $B \subseteq Y$)

and $\because f^{-1}(B^o) = (f^{-1}(B))^o \Rightarrow (f^{-1}(B))^o \subseteq f^{-1}(B^o) \Rightarrow f^{-1}$ is cont.

(by theory f is cont $\Leftrightarrow (f(B))^o \subseteq f(B^o)$; $B \subseteq X$ and f replace by f^{-1})

$\therefore$ f is Home.

($\Leftarrow$) Suppose that f is Home., to prove $f^{-1}(B^o) = (f^{-1}(B))^o$

$\because$ f is home. $\Rightarrow$ f is cont. and f^{-1} is cont.

$\Rightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o$ and $(f^{-1}(B))^o \subseteq f^{-1}(B^o)$

$\Rightarrow f^{-1}(B^o) = (f^{-1}(B))^o$

Definition : Topological Property

A property "P" of a topological space (X, τ) is called a **topological property** iff every topological space (Y, τ') homeomorphic to (X, τ) also has the same property. i.e., if $(X, \tau) \cong (Y, \tau')$ and (X, τ) has a property "P", then (Y, τ') has the same property and vise versa.

(2) ($\Leftarrow$) Suppose that $f^{-1}(B^o) = (f^{-1}(B))^o$, to prove f Home.,

$\because$ f is bij., we must prove f is cont. and f^{-1} is cont.

$\because f^{-1}(B^o) = (f^{-1}(B))^o \Rightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o \Rightarrow$ f is cont.

(by theory f is cont $\Leftrightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o$; $B \subseteq Y$)

and $\because f^{-1}(B^o) = (f^{-1}(B))^o \Rightarrow (f^{-1}(B))^o \subseteq f^{-1}(B^o) \Rightarrow f^{-1}$ is cont.

(by theory f is cont $\Leftrightarrow (f(B))^o \subseteq f(B^o)$; $B \subseteq X$ and f replace by f^{-1})

$\therefore$ f is Home.

($\Leftarrow$) Suppose that f is Home., to prove $f^{-1}(B^o) = (f^{-1}(B))^o$

$\because$ f is home. $\Rightarrow$ f is cont. and f^{-1} is cont.

$\Rightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o$ and $(f^{-1}(B))^o \subseteq f^{-1}(B^o)$

$\Rightarrow f^{-1}(B^o) = (f^{-1}(B))^o$

Definition : Topological Property

A property "P" of a topological space (X, τ) is called a **topological property** iff every topological space (Y, τ') homeomorphic to (X, τ) also has the same property. i.e., if $(X, \tau) \cong (Y, \tau')$ and (X, τ) has a property "P", then (Y, τ') has the same property and vise versa.

Example: Show that $[a,b] \cong [0,1]$, $a, b \in \mathbb{R}$.

Solution: Let X = [a,b] and Y = [0,1] with the usual topology

To show that X and Y are homeomorphic, i.e., to find a homeomorphism from X onto Y.

Let $y = f(x) = c_1x + c_2$, such that if f maps x = 0 onto y = a and maps x = 1 onto y = 1

i.e., to find c_1 and c_2 which make f a homeomorphism

Now, if x = 0 we must have y = a and substitute in f, we get $a = c_1 * 0 + c_2 \Rightarrow c_2 = a$

If x = b we must have y = b and substitute in f, we get $b = c_1 * 1 + c_2 \Rightarrow b = c_1 + a$

$\Rightarrow c_1 = b - a$

Hence $y = f(x) = (b - a)\, x + a$ (which is a linear function)

Now to show that f is a homomorphism function between X onto Y

$\because$ f is a linear function, then f is continuous and it is easily shown that f is 1-1 and onto

$\therefore f^{-1}$ exists which is defined by $x = f^{-1}(y) = \frac{y-a}{b-a} = \frac{1}{b-a}y - \frac{a}{b-a}$, which is also a linear function in y and hence it is continuous

$\therefore$ f is a homomorphism function and so $[a,b] \cong [0,1]$

Example:

Let $X = (-1, 1)$. The function $f : X \to \mathbf{R}$ defined by $f(x) = \tan \frac{1}{2}\pi x$ is one-one, onto and continuous. Furthermore, the inverse function f^{-1} is also continuous. Hence the real line $\mathbf{R}$ and the open interval $(-1, 1)$ are homeomorphic.
Hence *length* is not a topological property since X and $\mathbf{R}$ have different lengths, and *boundedness* is not a topological property since X is bounded but $\mathbf{R}$ is not.

Compact Spaces

Definition : Cover & Finite Cover & Open (resp., Closed) Cover

Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a family of subsets of the space (X, τ). We called the family $\{A_\alpha\}_{\alpha\in\Lambda}$ **cover** of X iff X equal the union of elements of the family $\{A_\alpha\}_{\alpha\in\Lambda}$.

$$(\text{i.e., } X = \bigcup_{\alpha\in\Lambda} A_\alpha)$$

If $\{A_\alpha\}_{\alpha\in\Lambda}$ is finite and cover X, then $\{A_\alpha\}_{\alpha\in\Lambda}$ is called a **finite cover** of X.

If each A_α, $\alpha\in\Lambda$, is open (resp., closed) in X and $\{A_\alpha\}_{\alpha\in\Lambda}$ cover X, then $\{A_\alpha\}_{\alpha\in\Lambda}$ is called an **open (resp., closed) cover** of X.

Definition : Subcover

Let $C = \{A_\alpha\}_{\alpha\in\Lambda}$ be a cover of X and $\{B_i\}_{i\in\Lambda}$ be a sub family of C and cover X, then $\{B_i\}_{i\in\Lambda}$ is called **subcover** from C.

Definition : Compact Space

A space X is called **compact** iff each open cover of X has a finite subcover for X.
i.e.,

$$X \text{ is compact} \Leftrightarrow \forall\ C = \{U_\alpha\}_{\alpha\in\Lambda}\ ;\ U_\alpha \in \tau\ \forall\ \alpha\ \wedge\ X = \bigcup_{\alpha\in\Lambda} U_\alpha$$
$$\Rightarrow \exists\ \alpha_1, \alpha_2, \dots, \alpha_n\ ; X = \bigcup_{i=1}^{n} U_{\alpha_i}.$$
$$X \text{ is not compact} \Leftrightarrow \exists\ C = \{U_\alpha\}_{\alpha\in\Lambda}\ ;\ U_\alpha \in \tau\ \forall\ \alpha\ \wedge\ X = \bigcup_{\alpha\in\Lambda} U_\alpha$$
$$\Rightarrow \nexists\ \alpha_1, \alpha_2, \dots, \alpha_n\ ; X = \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Example : Take $X = \mathbb{R}$ and $\tau = \{\mathbb{R}, \phi, \mathbb{Q}, \text{Irr}\}$
The open set in τ are $\mathbb{R}$, ϕ, $\mathbb{Q}$, Irr.
Take, $C_1 = \{\mathbb{Q}, \text{Irr}\}$ is open cover for $\mathbb{R}$ (i.e., $\mathbb{R} = \mathbb{Q} \bigcup \text{Irr}$) and it's a finite subcover of $\mathbb{R}$, so this cover satisfy the definition of compact space.
Now, we introduce all open cover for $\mathbb{R}$ as follow :

$$C_2 = \{\mathbb{R}, \phi, \mathbb{Q}\},\ C_3 = \{\mathbb{R}, \phi, \text{Irr}\},\ C_4 = \{\mathbb{R}, \mathbb{Q}, \text{Irr}\},\ C_5 = \{\phi, \mathbb{Q}, \text{Irr}\},$$

$$C_6 = \{\mathbb{R}, \phi\}, C_7 = \{\mathbb{R}, \mathbb{Q}\}, C_8 = \{\mathbb{R}, Irr\}, C_9 = \{\mathbb{R}, \phi, \mathbb{Q}, Irr\} = \tau$$

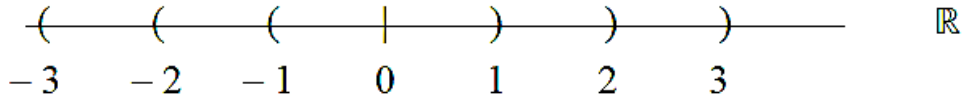
and every cover of them has a finite subcover, hence $(\mathbb{R}, \tau)$ is compact.

Remark : If we want to show that the space is not compact it's enough to give one open cover, but has not finite subcover. The following example show this :

Example : Is $(\mathbb{R}, \tau_u)$ compact space ???

Answer : **No**

Take the open cover $C = \{(-n, n) \ ; \ n \in \mathbb{N}\}$ for $\mathbb{R}$ (i.e., $\mathbb{R} = \bigcup_{i=1}^{n}(-n, n)$).



Notes that the open cover C has no finite subcover, because if we assumption there exists a finite open interval cover $\mathbb{R}$, then their union is the large interval, for example $(-m, m)$; $m \in \mathbb{N}$, this means $\mathbb{R} = (-m, m)$, $m \neq \infty$ and this contradiction!!!

Example : Show that $(\mathbb{N}, \tau_{cof})$ is compact space.

Solution : Let $C = \{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover for $\mathbb{N}$, then

$$\mathbb{N} = \bigcup_{\alpha \in \Lambda} U_\alpha \ ; \ U_\alpha \in \tau_{cof} \ \forall \ \alpha \in \Lambda$$

since $U_\alpha \in \tau_{cof}$, then $\mathbb{N} - U_\alpha$ if finite , for all $U_\alpha \in \tau_{cof}$

take arbitrary set say U_{α_n}, then $\mathbb{N} - U_{\alpha_n}$ if finite,

let $\mathbb{N} - U_{\alpha_n} = \{x_1, \ldots, x_{n-1}\}$; $x_1, \ldots, x_{n-1} \in \mathbb{N}$

this means that U_{α_n} contains all natural numbers excepts $x_1, \ldots, x_{n-1}$

take another set contains x_1 say U_{α_1} and set contains x_2 say U_{α_2} .... etc set contains x_{n-1} say $U_{\alpha_{n-1}}$. So we have n set which are U_{α_n} ,....., U_{α_1} such that $\mathbb{N} = \bigcup_{i=1}^{n} U_{\alpha_i}$.

therefore, the open cover $C = \{U_\alpha\}_{\alpha \in \Lambda}$ has a finite subcover $\{U_{\alpha_i}\}_{i=1}^{n}$, hence $(\mathbb{N}, \tau_{cof})$ is compact.

Definition : Compact Subspace

Let (X, τ) be a topological space and W be a subspace of X. We called a space W is **compact space** iff every open cover from X cover W has a finite subcover. i.e.,

$$W \text{ is compact} \Leftrightarrow \forall \ \{U_\alpha\}_{\alpha \in \Lambda} \ ; U_\alpha \in \tau \ \forall \ \alpha \ \wedge \ W \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$$
$$\Rightarrow \exists \ \alpha_1, \alpha_2, \ldots, \alpha_n \ ; W \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Theorem : (Heine-Borel Theorem)

The subset $A \subseteq X$ is compact iff A is closed and bounded.

Remarks : The previous theorem is one of theorems which study in mathematical analysis which is specific to subsets of Euclidean space $\mathbb{R}$ with usual topology and special case in $(\mathbb{R}, \tau_u)$. So, every subset of $\mathbb{R}$ is compact iff its closed and bounded.

Example : In $(\mathbb{R}, \tau_u)$ show that any set from the following sets is compact or not by using Heine-Borel Theorem.

$A = (2, 3)$, , $B = [5, 7]$, $C = [-2, 1)$, $D = \mathbb{N}$, , $E = \{2, 3, 4\}$, $F = \mathbb{Q}$

Solution :

A not closed and bounded $\Rightarrow$ not compact.

B closed and bounded $\Rightarrow$ compact.

C not closed and bounded $\Rightarrow$ not compact.

D closed and not bounded $\Rightarrow$ not compact.

E closed and bounded $\Rightarrow$ compact.

F not closed and not bounded $\Rightarrow$ not compact.

In general in usual topology $(\mathbb{R}, \tau_u)$,

every closed intervals is compact sets.

every half closed (open) interval and open intervals is not compact sets.

every finite sets of points is compact sets.

$\mathbb{Q}$ and Irr not compact.

Definition : Hereditary Property

We call a property "P" of a space (X, τ) **hereditary property** iff every subspace of a space X have the same property.

Notes that if there exists at least one subspace not satisfy this property, then this property not hereditary property.

Remark : Compactness is not hereditary property. For example :

Example : Take X = [0,1] with induce topology from $(\mathbb{R}, \tau_u)$ and take W = (0, 1). Clear that $W \subseteq X$ and X is compact space, but W is not compact.

Theorem : If A and B are compact sets in a space (X, τ), then $A \bigcup B$ is compact set.

Proof : Let $C = \{U_\alpha\}_{\alpha \in \Lambda}$; $U_\alpha \in \tau \ \forall \ \alpha$ open cover of $A \bigcup B$.

To prove C has a finite subcover

$\because \ A \bigcup B \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha \Rightarrow A \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha \wedge B \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$

(since $A \subseteq A \bigcup B$, $B \subseteq A \bigcup B$)

$\therefore$ C cover of A and B, but A and B are compact

$\Rightarrow \exists\ \alpha_1, \dots, \alpha_n\ ; A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$ and $\exists\ \alpha_1, \dots, \alpha_m\ ; B \subseteq \bigcup_{j=1}^{m} U_{\alpha_j}$

$\Rightarrow A \bigcup B \subseteq \bigcup_{k=1}^{n+m} U_{\alpha_k}$

$\therefore$ C has finite subcovet for $A \bigcup B \Rightarrow A \bigcup B$ compact set.

Remark : If A and B are compact sets in a space (X, τ), then $A \bigcap B$ is not necessary compact set.

Theorem : The continuous image of compact space is compact. i.e.,
If $f : (X, \tau) \to (Y, \tau')$ is continuous function and X is compact space, then f(X) is compact.

Proof : Let $f : (X, \tau) \to (Y, \tau')$ be continuous and X compact space.
To prove, f(X) compact in Y
Let $C = \{V_\alpha\}_{\alpha \in \Lambda}$ open cover for f(X)

$\Rightarrow f(X) \subseteq \bigcup_{\alpha \in \Lambda} V_\alpha\ ; V_\alpha \in \tau'\ \forall\ \alpha \in \Lambda$

$\Rightarrow f^{-1}(f(X)) \subseteq f^{-1}(\bigcup_{\alpha \in \Lambda} V_\alpha)$

$\Rightarrow X \subseteq \bigcup_{\alpha \in \Lambda} f^{-1}(V_\alpha)$ (since $f^{-1}(f(X)) = X$ and

$f^{-1}(\bigcup_{\alpha \in \Lambda} A_\alpha) = \bigcup_{\alpha \in \Lambda} f^{-1}(A_\alpha)$)

Since f is continuous $\Rightarrow f^{-1}(V_\alpha) \in \tau\ \forall\ \alpha \in \Lambda$

$\Rightarrow \{f^{-1}(V_\alpha)\}_{\alpha \in \Lambda}$ is open cover for X

$\because$ X is compact $\Rightarrow \exists\ \alpha_1, \dots, \alpha_n\ ; X \subseteq \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$

$\Rightarrow f(X) \subseteq f(\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i}))$

$\Rightarrow = \bigcup_{i=1}^{n} f(f^{-1}(V_{\alpha_i}))$ (since $f(A \bigcup B) = f(A) \bigcup f(B)$)

$\Rightarrow f(X) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$ (since $f(f^{-1}(A)) \subseteq A$)

$\therefore$ f(X) compact set.

Theorem : Compactness is a topological property.
Proof : Let (X, τ) and (Y, τ') be topological space ; $X \cong Y$
Suppose that X is compact, To prove Y is compact
$\because X \cong Y \Rightarrow \exists\ f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f^{-1} continuous
$\because$ f continuous, onto and X compact $\Rightarrow f(X) = Y$ compact
(by theorem : The continuous image of compact space is compact)

Now, suppose that Y is compact, To prove X is compact

$\because$ f^{-1} continuous, onto and Y compact $\Rightarrow$ $f^{-1}(Y) = X$ compact

(by same theorem : The continuous image of compact space is compact)

Theorem : A **closed** subset of a compact space is compact.

Proof :

Let (X, τ) compact space and F closed set in X

To prove, F compact set

Let $C = \{U_\alpha\}_{\alpha\in\Lambda}$ open cover of F

$$\Rightarrow F \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha \; ; U_\alpha \in \tau \;\; \forall \alpha \in \Lambda$$

$\because$ $X = F \bigcup F^c \Rightarrow X = \bigcup_{\alpha\in\Lambda} U_\alpha \bigcup F^c$ (since $F \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$)

$\because$ $U_\alpha \in \tau \;\; \forall \alpha \in \Lambda \;\; \wedge \;\; F^c \in \tau$ (since F closed set)

$$\Rightarrow \{U_\alpha\}_{\alpha\in\Lambda} \bigcup \{F^c\} \text{ open cover of } X$$

$\because$ X compact $\Rightarrow \exists\, \alpha_1, \dots, \alpha_n \; ; X = (\bigcup_{i=1}^{n} U_{\alpha_i}) \bigcup F^c$

But, $F \subseteq X \Rightarrow F \subseteq (\bigcup_{i=1}^{n} U_{\alpha_i}) \bigcup F^c$

Since $F \bigcap F^c = \phi \Rightarrow F \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$

$\therefore$ F compact set.

Now, suppose that Y is compact, To prove X is compact

$\because$ f^{-1} continuous, onto and Y compact $\Rightarrow$ $f^{-1}(Y) = X$ compact

(by same theorem : The continuous image of compact space is compact)

Theorem : A **closed** subset of a compact space is compact.

Proof :

Let (X, τ) compact space and F closed set in X

To prove, F compact set

Let $C = \{U_\alpha\}_{\alpha\in\Lambda}$ open cover of F

$$\Rightarrow F \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha \; ; U_\alpha \in \tau \;\; \forall \alpha \in \Lambda$$

$\because$ $X = F \bigcup F^c \Rightarrow X = \bigcup_{\alpha\in\Lambda} U_\alpha \bigcup F^c$ (since $F \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$)

$\because$ $U_\alpha \in \tau \;\; \forall \alpha \in \Lambda \;\wedge\; F^c \in \tau$ (since F closed set)

$\Rightarrow$ $\{U_\alpha\}_{\alpha\in\Lambda} \bigcup \{F^c\}$ open cover of X

$\because$ X compact $\Rightarrow \exists\, \alpha_1, \dots, \alpha_n \; ; X = (\bigcup_{i=1}^{n} U_{\alpha_i}) \bigcup F^c$

But, $F \subseteq X \Rightarrow F \subseteq (\bigcup_{i=1}^{n} U_{\alpha_i}) \bigcup F^c$

Since $F \bigcap F^c = \phi \Rightarrow F \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$

$\therefore$ F compact set.

Connected Spaces

Definition: Let (X,τ) be a topological space, then two subsets A and B of X are said to be ***separated*** if $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$.

Remark: Every separated sets are disjoint, while the converse is not true in general, since if A and B are two separated sets of a topological space (X,τ), then by definition $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$.

Since $A \subseteq \overline{A}$, then $A \cap B \subseteq \overline{A} \cap B = \varnothing \Rightarrow A \cap B = \varnothing \Rightarrow$ A and B are disjoint

The converse is not true and for this consider as an example two sets $A = (0,1]$ and $B = (1,2)$, then $\overline{A} = [0,1]$, $\overline{B} = [1,2]$. Then $A \cap \overline{B} = (0,1] \cap [1,2] \neq \varnothing$

$\therefore$ A and B are not separated, but they are disjoint sets

Definition: Let (X,τ) be a topological space, then a subset A of X is said to be ***disconnected*** if there exists open subsets G and H of X, such that $A \cap G$ and $A \cap H$ are disjoint non-empty sets whose union is A. In this case $G \cup H$ is called the ***disconnection*** of A. A set A is ***connected*** if it is not disconnected.

Now, if A is disconnected, then

$A = (A \cap G) \cup (A \cap H)$ iff $A \subset G \cup H$, and

$\varnothing = (A \cap G) \cap (A \cap H)$ iff $G \cap H \subset A^c$

Therefore, $G \cup H$ is a disconnection of A if and only if:

$A \cap G \neq \varnothing$, $A \cap H \neq \varnothing$, $A \subset G \cup H$ and $G \cap H \subset A^c$

Remark: The empty set $\varnothing$ and the singleton set {p} are always connected. (H.W.)

Example:

Consider the following topology on $X = \{a, b, c, d, e\}$:

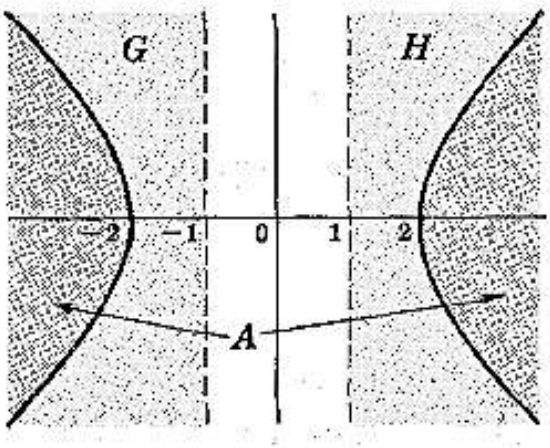
$$\mathcal{T} = \{X, \emptyset, \{a, b, c\}, \{c, d, e\}, \{c\}\}$$

Now $A = \{a, d, e\}$ is disconnected. For let $G = \{a, b, c\}$ and $H = \{c, d, e\}$; then $A \cap G = \{a\}$ and $A \cap H = \{d, e\}$ are non-empty disjoint sets whose union is A (Observe that G and H are not disjoint.)

Example:

The following subset of the plane $\mathbb{R}^2$ is disconnected:

$$A = \{\langle x, y\rangle : x^2 - y^2 \geq 4\}$$



For the two open half-planes

$$G = \{(x, y) : x < -1\} \quad \text{and} \quad H = \{(x, y) : x > 1\}$$

form a disconnection of A as indicated in the diagram above.

Remarks:

(1) If A and B are non-empty separated sets, then A ∪ B is disconnected.

(2) If G ∪ H is a disconnection of A, then A ∩ G and A ∩ H are separated sets.

Proof:

(1) Since A and B are separated, $A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset$. Let $G = \bar{B}^c$ and $H = \bar{A}^c$. Then G and H are open and

$$(A \cup B) \cap G = A \quad \text{and} \quad (A \cup B) \cap H = B$$

are non-empty disjoint sets whose union is $A \cup B$. Thus G and H form a disconnection of $A \cup B$, and so $A \cup B$ is disconnected.

(2) If $A \cap G$ and $A \cap H$ are disjoint; hence we need only show that each set contains no accumulation point of the other, Let p be an accumulation point of $A \cap G$, and suppose $p \in A \cap H$. Then H is an open set containing p and so H contains a point of $A \cap G$ distinct from p, i.e. $(A \cap G) \cap H \neq \emptyset$. But

$$(A \cap G) \cap (A \cap H) = \emptyset = (A \cap G) \cap H$$

Accordingly, $p \notin A \cap H$.

Similarly, if p is an accumulation point of $A \cap H$, then $p \notin A \cap G$. Thus $A \cap G$ and $A \cap H$ are separated sets.

Theorem: A set A is connected if and only if A is not a union of two non-empty separated sets.

Proof:

We will prove the negation of the theorem, i.e., we will prove equivalently that A is disconnected if and only if A is a union of two non-empty separated sets.

Suppose A is disconnected ⇒ there exists disconnection G ∪ H of A

$\Rightarrow A = (A \cap G) \cup (A \cap H)$ and $\varnothing = (A \cap G) \cap (A \cap H)$

Where $A \cap G \neq \varnothing$ and $A \cap H \neq \varnothing$

By Remark (2), $A \cap G$ and $A \cap H$ are separated sets

Conversely, if A is the union of two non-empty separated sets, then by Remark (1) A is disconnected.

Remark:

Let $G \cup H$ be a disconnection of A and let B be a connected subset of A, then either $B \cap G = \varnothing$ or $B \cap H = \varnothing$, and so either $B \subset G$ or $B \subset H$.

Proof:

Since $G \cup H$ is a disconnection of A, then $A \subset G \cup H$ and also $B \subset A$

$\Rightarrow B \subset G \cup H$

Also, $G \cap H \subset A^c \Rightarrow G \cap H \subset B^c$

Thus, if both $B \cap G \neq \varnothing$ and $B \cap H \neq \varnothing$, then $G \cup H$ is a disconnection of B, i.e., B is disconnected (which is a contradiction c!)

Hence, ether $B \cap G = \varnothing$ or $B \cap H = \varnothing$ or both.

Theorem:

If A and B are connected sets which are not separated, then $A \cup B$ is connected.

Proof:

Suppose $A \cup B$ is disconnected and suppose $G \cup H$ is a disconnection of $A \cup B$. Since A is a connected subset of $A \cup B$, either $A \subset G$ or $A \subset H$. Similarly, either $B \subset G$ or $B \subset H$.

Now if $A \subset G$ and $B \subset H$ (or $B \subset G$ and $A \subset H$), then,

$$(A \cup B) \cap G = A \quad \text{and} \quad (A \cup B) \cap H = B$$

are separated sets. But this contradicts the hypothesis; hence either $A \cup B \subset G$ or $A \cup B \subset H$, and so $G \cup H$ is not a disconnection of $A \cup B$. In other words, $A \cup B$ is connected.

Next, we will study the connectedness of the whole topological space.

Definition: A pair of open sets {U,V} in a topological space X is called a sepearation (disconnection) of X if $X = U \cup V$, $U \neq \emptyset$, $V \neq \emptyset$ and $U \cap V = \emptyset$.

Definition: A topological space X is called disconnected if it has a separation, and connected if no separation exists.

Another useful alternative characterization of connectedness is the following property stated and proved in the next theorem:

Theorem : (X, τ) is connected space iff the only subsets of the space X which are open and closed are X and ϕ.

Proof : ($\Rightarrow$) Suppose that X is connected

To prove, if $A \subseteq X$, $A, A^c \in \tau$, then $A = X$ or $A = \phi$.

Suppose that $A, A^c \in \tau$ and $A \neq X$ and $A \neq \phi$

$\Rightarrow X = A \bigcup A^c \ \wedge \ A, A^c \in \tau \ \wedge \ A \bigcap A^c = \phi \ , A \neq \phi \neq A^c$ (since $A \neq X$)

$\Rightarrow$ X disconnected C!! Contradiction !!

$\therefore$ if $A \subseteq X$, $A, A^c \in \tau$, then $A = X$ or $A = \phi$.

($\Leftarrow$) Suppose that, if $A \subseteq X$, $A, A^c \in \tau$, then $A = X$ or $A = \phi$, i.e., $\tau \bigcap \mathcal{F} = \{X, \phi\}$.

To prove X is connected

Suppose that X is disconnected

$\Rightarrow X = U \bigcup V \ ; U, V \in \tau \ , U \bigcap V = \phi \ \wedge \ U \neq \phi \neq V$

$\Rightarrow \ U = V^c \ \wedge \ V = U^c \ \Rightarrow U, V \in \mathcal{F}$

$\Rightarrow U, V \in \tau \bigcap \mathcal{F}$ C!! Contradiction !!

Since U and V are open and closed and not equal X and ϕ.

$\therefore$ X connected space.

Examples:

1. *$(X, \mathcal{T}_{ind})$ is connected, since the only clopen subsets of X are $\emptyset$ and X.*

2. $(X, \mathcal{T}_{dis})$ *with* $cardX > 1$ *is disconnected, since* $\{x\} \subsetneq X$ *is clopen or* $\{\{x\}, X \setminus \{x\}\}$ *is a separation of* X.
3. *Any singleton* $D = \{x\}$ *in a topological space* X *is connected, since* $\mathcal{T}_D = \{\emptyset, D\}$ *is the indiscrete topology.*
4. *Let* $(X, \mathcal{T}) = (\mathbb{R}, \mathcal{T}_{us})$. *The set* $A = \{-1, 1\}$ *is disconnected, since* $\mathcal{T}_A$ *is the discrete topology.*
5. $(X, \mathcal{T}_{sicr})$ *is connected, since the only clopen subsets of* X *are* $\emptyset$ *and* X.
6. $(\mathbb{R}, \mathcal{T}_{cof})$ *is connected. If it it admits a separation* $\{U, V\}$, *then* $U = V^c$ *a contradiction.*
7. $(\mathbb{R}, \mathcal{T}_{up})$ *is disconnected, since* $(1, 2]$ *is clopen.*
8. $\mathbb{Q}$ *is a disconnected subset of* $\mathbb{R}$, *since the open sets* $\{(-\infty, \sqrt{2}) \cap Q, (\sqrt{2}, \infty) \cap Q\}$ *is a separation of* $\mathbb{Q}$.
9. $A = (0, 1] \cup (2, 3)$ *and* $B = \mathbb{R}^*$ *are disconnected subsets of* $\mathbb{R}$.

Theorem : (X, τ) is connected space iff the only subsets of the space X which have empty boundary sets are X and ϕ. i.e.,

$$[X \text{ is connected } \Leftrightarrow (\phi \neq A \subsetneq X \Rightarrow A^{bd} \neq \phi)]$$

on the other hand :

$$[X \text{ is connected } \Leftrightarrow (A^{bd} = \phi \Rightarrow A = \phi \vee A = X)]$$

Proof : ($\Rightarrow$) Suppose that X is connected

To prove, if $A^{bd} = \phi \Rightarrow A = \phi \vee A = X$.

Suppose that $A \subseteq X$ and $A^{bd} = \phi \wedge A \neq \phi \wedge A \neq X$

$\Rightarrow \phi \subseteq A \Rightarrow A^{bd} \subseteq A \Rightarrow A \in \mathcal{F}$ (by theorem : $A \in \mathcal{F} \Leftrightarrow A^{bd} \subseteq A$)

where $\mathcal{F}$ is the set of all closed subsets of X

On the other hand :

$\phi \bigcap A = \phi \Rightarrow A^{bd} \bigcap A = \phi \Rightarrow A \in \tau$ (by theorem : $A \in \tau \Leftrightarrow A^{bd} \bigcap A = \phi$)

$\Rightarrow A, A^c \in \tau$ and $A \neq \phi \wedge A \neq X$

$\Rightarrow$ X disconnected

(by theorem : (X, τ) is connected space iff the only subsets of the space X which are open and closed are X or ϕ)

(this contradiction with hypotheses : X is connected $\Rightarrow A^{bd} \neq \phi$ if $\phi \neq A \neq X$)

$\therefore$ if $A^{bd} = \phi \Rightarrow A = \phi \vee A = X$.

($\Leftarrow$) Suppose that, if $A^{bd} = \phi \Rightarrow A = \phi \vee A = X$.

To prove X is connected

Suppose that X is disconnected

$\Rightarrow X = U \bigcup V \; ; U, V \in \tau \; , U \bigcap V = \phi \wedge U \neq \phi \neq V$

$\Rightarrow U = V^c \wedge V = U^c \Rightarrow U, V \in \mathcal{F}$

$\Rightarrow U^{bd} = \phi \wedge V^{bd} = \phi$ (by theorem : $A, A^c \in \tau \Leftrightarrow A^{bd} = \phi$)

(this contradiction since $U^{bd} = \phi$, but $U \neq X$ and $U \neq \phi$)

$\therefore$ X connected space.

Remark : If (X, τ) is topological space and (W, τ_W) is a subspace of X, then the space W being disconnected or connected not directly relation by X and the open sets in X, but dependent on the open sets in W. i.e., its dependent on τ_W so that : W is connected space iff there exist two open disjoint nonempty sets A and B in W such that $A \bigcup B = W$.

i.e., W is disconnected $\Leftrightarrow W = A \bigcup B \; ; \; A, B \in \tau_W \; , \; A \bigcap B = \phi \; , A \neq \phi \neq B$.

The space (W, τ_W) is **connected** iff it is not disconnected.

W is connected $\Leftrightarrow W \neq A \bigcup B \; ; \; A, B \in \tau_W \; , \; A \bigcap B = \phi \; , A \neq \phi \neq B$.

Remark : The property of being a connected space is not a hereditary property as the following example show that :

Example : Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1, 2\}, \{1, 3\}, \{1\}\}$. Let $W \subseteq X$; $W = \{2, 3\}$. Is W is connected space ??

Solution : Compute τ_W :

$$\tau_W = \{ W \bigcap U \; ; U \in \tau\} = \{W, \phi, \{2\}, \{3\},\}$$

Notes that $\tau_W = D$, then W is disconnected space but not connected since :

$W = \{2\} \bigcup \{3\}$ and $\{2\}, \{3\} \in \tau_W$ and $\{2\} \bigcap \{3\} = \phi$ and $\{2\} \neq \phi, \{3\} \neq \phi$.

Notes that X is connected space but not disconnected, while it's have disconnected subspace.

Theorem : continuous image of connected space is connected. i.e.,

If $f : (X, \tau) \rightarrow (Y, \tau')$ is continuous and onto function and X is connected, then Y is connected.

Proof : Suppose that Y is disconnected

$\Rightarrow \exists\ A, B \in \tau'\ ;\ A \bigcap B = \phi \wedge A \neq \phi \neq B \wedge Y = A \bigcup B$

$\Rightarrow f^{-1}(Y) = f^{-1}(A \bigcup B)$

$\Rightarrow X = f^{-1}(A) \bigcup f^{-1}(B)$ (since f onto $\Rightarrow f^{-1}(Y) = X \wedge f^{-1}(A \bigcup B) = f^{-1}(A) \bigcup f^{-1}(B)$)

$\Rightarrow f^{-1}(A), f^{-1}(B) \in \tau$ (since f continuous)

$\wedge\ f^{-1}(A) \bigcap f^{-1}(B) = f^{-1}(A \bigcap B) = f^{-1}(\phi) = \phi$

$\wedge\ A \neq \phi \wedge B \neq \phi \Rightarrow f^{-1}(A) \neq \phi \neq f^{-1}(B)$

$\therefore$ X is disconnected C! Contradiction

$\therefore$ Y is connected.

Remark : If $f : (X, \tau) \to (Y, \tau')$ is continuous and onto function and Y is connected space, then X not necessary connected space and the following example show that :

Example : Let $f : (\mathbb{R}, D) \to (\mathbb{R}, I)$; $f(x) = x \quad \forall\ x \in \mathbb{R}$.

Clear that f is continuous and onto function and $(\mathbb{R}, I)$ is connected, but $(\mathbb{R}, D)$ is not connected (disconnected).

Corollary: The property of being a connected space is a topological property.

Proof :

Let (X, τ) and (Y, τ') be topological space ; $X \cong Y$

$\because X \cong Y \Rightarrow \exists\ f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f^{-1} continuous

Suppose that X is connected, to prove Y is connected

$\because$ f continuous, onto and X connected $\Rightarrow f(X) = Y$ connected

(by theorem : continuous image of connected space is connected)

Now, suppose that Y is connected, to prove X is connected

$\because f^{-1}$ continuous, onto and Y connected $\Rightarrow f^{-1}(Y) = X$ connected

(by same theorem : continuous image of connected space is connected)

Example (H.W.): If we know the function $f : (\mathbb{R}, \tau_u) \to (Y, \tau)$ is continuous onto. Is Y connected space ?

Example (H.W.): If we know that $(\mathbb{R}, D) \cong (Y, \tau)$. Is Y connected space ?

Separation Axioms

Many properties of topological space X depend upon the distribution of the open sets in the space. Roughly speaking, a space is more likely to be separable, or first or second countable, if there are few open sets; in the other hand, an arbitrary function on X to some topological space is more likely to be continuous, or sequence to have a unique limit, if the space has many open sets.

The separation axioms of Alexandroff and Hopf, discussed in this section, postulate or discuss the existence of enough open sets.

Definition T_0 – Space:

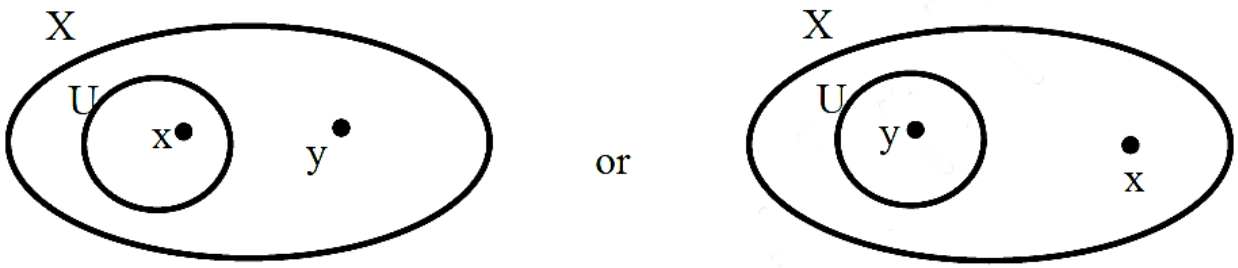
Let (X, τ) be a topological space. Then the space $(X, \tau)$is called **T_0 – space** iff for each pair of distinct points $x, y \in X$, there is either an open set containing x but not y or an open set containing y but not x. i.e.,

X is T_0 – space $\Leftrightarrow \forall\, x, y \in X \;;\; x \neq y; \exists\, U \in \tau \;; (x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)$.

If (X, τ) is not T_0 – space, we define,

X is not T_0 – space $\Leftrightarrow \exists\, x, y \in X \;;\; x \neq y \;\; \forall\, U \in \tau \;; (x, y \in U) \vee (x, y \notin U)$.

The following figure show the definition of T_0 – Space :



Example : Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1\}, \{1, 2\}\}$. Is (X, τ), T_0 – space ?

Solution: We must test the definitionfor and two points in X satisfies the definition or not

$1 \neq 2 \Rightarrow \exists$ open set $\{1\} \in \tau \;;\; 1 \in \{1\} \wedge 2 \notin \{1\}$

$1 \neq 3 \Rightarrow \exists$ open set $\{1\} \in \tau \;;\; 1 \in \{1\} \wedge 3 \notin \{1\}$

$2 \neq 3 \Rightarrow \exists$ open set $\{1, 2\} \in \tau \;;\; 2 \in \{1, 2\} \wedge 3 \notin \{1, 2\}$

$\therefore$ The definition is satisfied $\Rightarrow (X, \tau)$ is T_0 – space.

Example : Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$. Is (X, τ), T_0 – space ?

Solution : Since $b \neq c \Rightarrow \nexists$ open set containing b but not c or an open set containing c but not b.

Clear that X containing b, c and ϕ open set not containing b, c and $\{a\}$ open set not containing b, c.

Therefore, (X, τ) is not T_0 – space.

Example : Is $(\mathbb{N}, \tau_{cof})$ T_0 – space.

Solution : Since in general since $\mathbb{N}$ containing infinite numbers, then:

Let $n, m \in \mathbb{N}$; $n \neq m$, to prove $\exists$ open set $U \in \tau_{cof}$; $m \in U \wedge n \notin U$ or vice versa.

Take $U = \mathbb{N} \setminus \{n\} \Rightarrow m \in U \wedge n \notin U$

and $U \in \tau_{cof}$ (since $U^c = (\mathbb{N} \setminus \{n\})^c = \{n\}$ finite set by def. of τ_{cof})

Similarly, way we can take : $U = \mathbb{N} \setminus \{m\} \Rightarrow m \notin U \wedge n \in U$

and $U \in \tau_{cof}$ (since $U^c = (\mathbb{N} \setminus \{m\})^c = \{m\}$ finite set by def. of τ_{cof})

The two cases similar and satisfy the definition $\Rightarrow (\mathbb{N}, \tau_{cof})$ is T_0 – space.

Example : In the space (X, I) if X is any set containing more than one element, then (X, I) is not T_0 – space, since X contains more than one element we take $x, y \in X$; $x \neq y$ and $\nexists$ open set containing x but not y or an open set containing y but not x and ϕ open set not containing x, y.

Example : The space (X, D) is T_0 – space.

Solution : Let $x, y \in X$; $x \neq y$, then $\{x\} \in D$ i.e., $\{x\}$ open set (by definition D since $D = IP(X)$), hence $x \in \{x\}$ and $y \notin \{x\}$. We can take $\{y\}$ replace of $\{x\}$ and $y \in \{y\}$ and $x \notin \{y\}$. Therefore, (X, D) is T_0 – space.

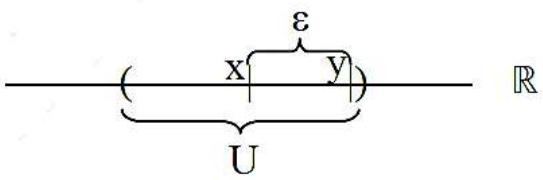
Example : The usual topological space $(\mathbb{R}, \tau_u)$ is T_0 – space.

Solution : Let $x, y \in \mathbb{R}$; $x \neq y$

Take $U = (x - \varepsilon, x + \varepsilon)$; $\varepsilon = |x - y|$

$\Rightarrow U \in \tau_u \wedge x \in U \wedge y \notin U$

$\therefore$ $(\mathbb{R}, \tau_u)$ is T_0 – space.



Theorem : (X, τ) is T_0 – space iff $\overline{\{x\}} \neq \overline{\{y\}}$ $\forall\, x, y \in X\,;\, x \neq y$.

i.e., (X, τ) is T_0 – space iff the closure of singleton sets is deference if the elements are deference.

Proof : ($\Rightarrow$) Suppose that X is T_0 – space, to prove $\overline{\{x\}} \neq \overline{\{y\}}$ $\forall$ $x, y \in X\,;\, x \neq y$

$\because$ X is T_0 – space and $x \neq y \Rightarrow \exists\, U \in \tau\,;\, (x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)$

Suppose that $(x \in U \wedge y \notin U) \Rightarrow (x \in U \wedge y \in X - U)$

$X - U$ closed set since U open set $\Rightarrow \{y\} \subseteq X - U$

$\Rightarrow \overline{\{y\}} \subseteq \overline{X - U} = X - U$

(since $X - U$ closed and $\overline{X - U} = X - U$)

$\Rightarrow \overline{\{y\}} \subseteq X - U \;\wedge\; x \in U$

$\Rightarrow \{x\} \not\subseteq X - U$

$\Rightarrow \overline{\{x\}} \not\subseteq X - U$

$\therefore\ \overline{\{x\}} \neq \overline{\{y\}}$

By similar way if we take $(x \notin U \wedge y \in U)$

($\Leftarrow$) suppose that $\overline{\{x\}} \neq \overline{\{y\}}$ $\forall$ $x \neq y \in X$, to prove X is T_0 – space

Suppose that X is not T_0 – space $\Rightarrow (\exists\, x, y \in X\,;\, \forall\, U \in \tau\,;\, x \in U \Rightarrow y \in U)$ (def of T_0 – space) (i.e., every open set containing x its containing y)

Let $z \in X$; $z \in \overline{\{x\}}$ ----------(*)

$\Rightarrow \forall\, U \in \tau\,;\, z \in U \wedge U \bigcap \{x\} \neq \phi$

(since by true : $z \in \overline{A} \;\Leftrightarrow\; \forall\, U \in \tau\,;\, z \in U \;\wedge\; U \bigcap A \neq \phi$)

But, $U \bigcap \{x\} \neq \phi \;\Rightarrow\; x \in U$ (since the only element in $\{x\}$ is x)

$\therefore$ every set contains z must contains x. So, we have the following two statements : every open set contains z must contains x and every open set contains x must contains y.

$\therefore$ every open set contains z must contains y.

$\Rightarrow \forall\, U \in \tau\,;\, z \in U \wedge U \bigcap \{y\} \neq \phi$

$\Rightarrow z \in \overline{\{y\}}$ ----------(*)

$\Rightarrow \forall\, z \in \overline{\{x\}} \;\Rightarrow\; z \in \overline{\{y\}} \;\Rightarrow \overline{\{x\}} \subseteq \overline{\{y\}}$

By similar way we prove $\overline{\{y\}} \subseteq \overline{\{x\}}$

$\therefore$ $\overline{\{x\}} = \overline{\{y\}}$ C!! contribution (since $\overline{\{x\}} \neq \overline{\{y\}}$)

$\therefore$ X is T_0 – space.

Theorem : The property of being a T_0 – space is a hereditary property.

Proof :

Let (X, τ) T_0– space and (W, τ_W) subspace of X, to prove (W, τ_W) is T_0– space

Let $x, y \in W$; $x \neq y \Rightarrow x, y \in X$ (since $W \subseteq X$)

$\because$ X is T_0– space $\Rightarrow \exists\, U \in \tau$; $(x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)$

$\Rightarrow U \bigcap W \in \tau_W$ (by def. of τ_W)

; $(x \in U \bigcap W \wedge y \notin U \bigcap W) \vee (x \notin U \bigcap W \wedge y \in U \bigcap W)$

$\therefore$ (W, τ_W) is T_0– space.

Theorem : The property of being a T_0 – space is a topological property.

Proof :

Let $(X, \tau) \cong (Y, \tau')$ and suppose that X is T_0– space, to prove Y is T_0– space

$\because$ $(X, \tau) \cong (Y, \tau') \Rightarrow \exists\, f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f^{-1} continuous

Let $y_1, y_2 \in Y$; $y_1 \neq y_2 \Rightarrow f^{-1}(y_1), f^{-1}(y_2) \in X$

$\because$ f onto function $\Rightarrow f^{-1}(y_1) \neq \phi$, $f^{-1}(y_2) \neq \phi$

$\because$ f 1-1 function $\Rightarrow \exists!\, x_1 \in X$; $f^{-1}(y_1) = x_1$ and $\exists!\, x_2 \in X$; $f^{-1}(y_2) = x_2$

and $x_1 \neq x_2$ and $x_1, x_2 \in X$

$\because$ X is T_0– space $\Rightarrow \exists\, U \in \tau$; $(x_1 \in U \wedge x_2 \notin U) \vee (x_1 \notin U \wedge x_2 \in U)$

$\because$ f^{-1} is cont. or f open $\Rightarrow f(U) \in \tau'$; $(f(x_1) \in f(U) \wedge f(x_2) \notin f(U))$

$\vee$ $(f(x_1) \notin f(U) \wedge f(x_2) \in f(U))$

$\therefore$ Y is T_0– space

Similarly, we prove, if Y is T_0– space, then X is T_0– space.

Theorem : Let (X, τ) and (Y, τ') be two topological spaces. Then the product space $X \times Y$ is a T_0 – space iff each X and Y are T_0 – space.

Proof :

($\Rightarrow$) Suppose that $X \times Y$ is a T_0– space, to prove that X and Y are T_0– space

Let $x_1, x_2 \in X$; $x_1 \neq x_2$ and $y_1, y_2 \in Y$; $y_1 \neq y_2$

$\Rightarrow (x_1, y_1), (x_2, y_2) \in X \times Y$; $(x_1, y_1) \neq (x_2, y_2)$

$\because$ $X \times Y$ is a T_0– space $\Rightarrow \exists$ an open set $U \times V \in \tau_{X \times Y}$;

$((x_1, y_1) \in U \times V \wedge (x_2, y_2) \notin U \times V) \vee ((x_1, y_1) \notin U \times V \wedge (x_2, y_2) \in U \times V)$

$\Rightarrow \exists\, U \in \tau$; $(x_1 \in U \wedge x_2 \notin U) \vee (x_1 \notin U \wedge x_2 \in U) \Rightarrow$ X is a T_0– space

and $\exists\, V \in \tau'$; $(y_1 \in V \wedge y_2 \notin V) \vee (y_1 \notin V \wedge y_2 \in V) \Rightarrow$ Y is a T_0– space.

($\Leftarrow$) Suppose that X and Y are T_0 – space, to prove X×Y is a T_0 – space

Let $(x_1, y_1), (x_2, y_2) \in X \times Y$; $(x_1, y_1) \neq (x_2, y_2)$

By def. product space $\Rightarrow (x_1, x_2 \in X \wedge x_1 \neq x_2) \wedge (y_1, y_2 \in Y \wedge y_1 \neq y_2)$

$\because$ X is a T_0 – space $\Rightarrow \exists\, U \in \tau$; $(x_1 \in U \wedge x_2 \notin U) \vee (x_1 \notin U \wedge x_2 \in U)$

$\because$ Y is a T_0 – space $\Rightarrow \exists\, V \in \tau'$; $(y_1 \in V \wedge y_2 \notin V) \vee (y_1 \notin V \wedge y_2 \in V)$

$\Rightarrow \exists\, U \times V$ is a basic open set ; $((x_1, y_1) \in U \times V \wedge (x_2, y_2) \notin U \times V)$

$\vee ((x_1, y_1) \notin U \times V \wedge (x_2, y_2) \in U \times V)$

$\therefore$ X×Y is a T_0 – space.

Definition T_1 – Space:

Let (X, τ) be a topological space. Then the space (X, τ)is called **T_1 – space** iff for each pair of distinct points x, y ∈ X, there exists an open set in X containing x but not y, and an open set in X containing y but not x. i.e.,

X is T_1 – space $\Leftrightarrow \forall\, x, y \in X$; $x \neq y\ \exists\, U, V \in \tau$; $(x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V)$.

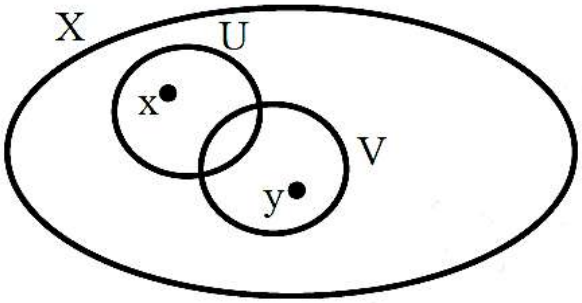
If (X, τ) is not T_1 – space, we define,

X is not T_1 – space $\Leftrightarrow \exists\, x, y \in X$; $x \neq y\ \forall\, U, V \in \tau$

; $(x \in U \wedge y \in U) \vee (x \notin U \wedge y \notin U)$

$(x \in V \wedge y \in V) \vee (x \notin V \wedge y \in V)$.

The following figure illustrate the definition of T_1 – space :



Remark : Every T_1 – space is T_0 – space (i.e., $T_1 \Rightarrow T_0$). But the reverse implications does not hold (i.e., $T_0 \not\Rightarrow T_1$) and the following example show that :

Example : Let (X, τ) be a topological space such that X = {1, 2, 3} and τ = {X, ϕ, {1}, {1, 2}}.

Solution : Clear (X, τ) is T_0 – space

But, (X, τ) is not T_1 – space, since 2 ≠ 3 and ∃ open set {1, 2} in X containing 2 but not 3, but ∄ open set in X containing 3 but not 2, since the only open set containing 3 is X and X containing 2 too.

Separation Axioms

Many properties of topological space X depend upon the distribution of the open sets in the space. Roughly speaking, a space is more likely to be separable, or first or second countable, if there are few open sets; in the other hand, an arbitrary function on X to some topological space is more likely to be continuous, or sequence to have a unique limit, if the space has many open sets.

The separation axioms of Alexandroff and Hopf, discussed in this section, postulate or discuss the existence of enough open sets.

Definition T_0 – Space:

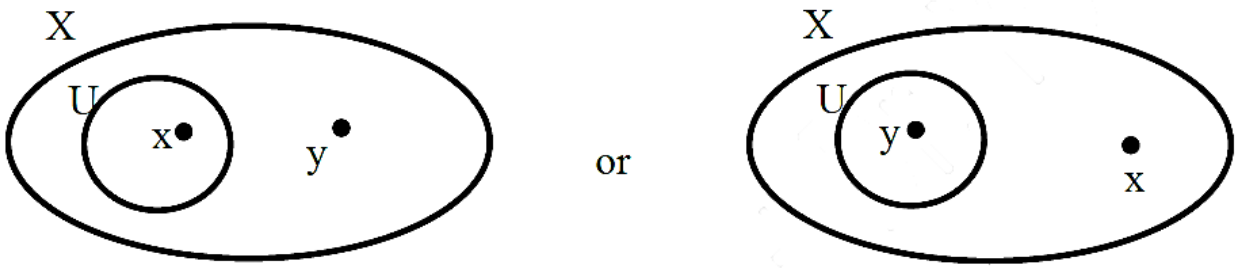
Let (X, τ) be a topological space. Then the space $(X, \tau)$is called **T_0 – space** iff for each pair of distinct points $x, y \in X$, there is either an open set containing x but not y or an open set containing y but not x. i.e.,

X is T_0 – space $\Leftrightarrow \forall\, x, y \in X \;;\; x \neq y; \exists\, U \in \tau \;; (x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)$.

If (X, τ) is not T_0 – space, we define,

X is not T_0 – space $\Leftrightarrow \exists\, x, y \in X \;;\; x \neq y \;\; \forall\, U \in \tau \;; (x, y \in U) \vee (x, y \notin U)$.

The following figure show the definition of T_0 – Space :



Example : Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1\}, \{1, 2\}\}$. Is (X, τ), T_0 – space ?

Solution: We must test the definitionfor and two points in X satisfies the definition or not

$1 \neq 2 \Rightarrow \exists$ open set $\{1\} \in \tau \;;\; 1 \in \{1\} \wedge 2 \notin \{1\}$

$1 \neq 3 \Rightarrow \exists$ open set $\{1\} \in \tau \;;\; 1 \in \{1\} \wedge 3 \notin \{1\}$

$2 \neq 3 \Rightarrow \exists$ open set $\{1, 2\} \in \tau \;;\; 2 \in \{1, 2\} \wedge 3 \notin \{1, 2\}$

$\therefore$ The definition is satisfied $\Rightarrow (X, \tau)$ is T_0 – space.

Example : Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$. Is (X, τ), T_0 – space ?

Solution : Since $b \neq c \Rightarrow \nexists$ open set containing b but not c or an open set containing c but not b.

Clear that X containing b, c and ϕ open set not containing b, c and $\{a\}$ open set not containing b, c.

Therefore, (X, τ) is not T_0 – space.

Example : Is $(\mathbb{N}, \tau_{cof})$ T_0 – space.

Solution : Since in general since $\mathbb{N}$ containing infinite numbers, then:

Let $n, m \in \mathbb{N}$; $n \neq m$, to prove $\exists$ open set $U \in \tau_{cof}$; $m \in U \wedge n \notin U$ or vice versa.

Take $U = \mathbb{N} \setminus \{n\} \Rightarrow m \in U \wedge n \notin U$

and $U \in \tau_{cof}$ (since $U^c = (\mathbb{N} \setminus \{n\})^c = \{n\}$ finite set by def. of τ_{cof})

Similarly, way we can take : $U = \mathbb{N} \setminus \{m\} \Rightarrow m \notin U \wedge n \in U$

and $U \in \tau_{cof}$ (since $U^c = (\mathbb{N} \setminus \{m\})^c = \{m\}$ finite set by def. of τ_{cof})

The two cases similar and satisfy the definition $\Rightarrow (\mathbb{N}, \tau_{cof})$ is T_0 – space.

Example : In the space (X, I) if X is any set containing more than one element, then (X, I) is not T_0 – space, since X contains more than one element we take $x, y \in X$; $x \neq y$ and $\nexists$ open set containing x but not y or an open set containing y but not x and ϕ open set not containing x, y.

Example : The space (X, D) is T_0 – space.

Solution : Let $x, y \in X$; $x \neq y$, then $\{x\} \in D$ i.e., $\{x\}$ open set (by definition D since $D = IP(X)$), hence $x \in \{x\}$ and $y \notin \{x\}$. We can take $\{y\}$ replace of $\{x\}$ and $y \in \{y\}$ and $x \notin \{y\}$. Therefore, (X, D) is T_0 – space.

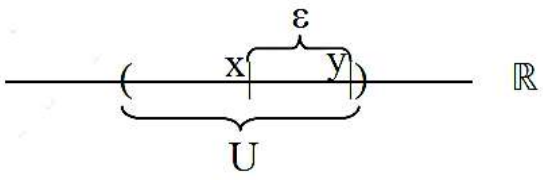
Example : The usual topological space $(\mathbb{R}, \tau_u)$ is T_0 – space.

Solution : Let $x, y \in \mathbb{R}$; $x \neq y$

Take $U = (x - \varepsilon, x + \varepsilon)$; $\varepsilon = |x - y|$

$\Rightarrow U \in \tau_u \wedge x \in U \wedge y \notin U$

$\therefore$ $(\mathbb{R}, \tau_u)$ is T_0 – space.



Theorem : (X, τ) is T_0 – space iff $\overline{\{x\}} \neq \overline{\{y\}}$ $\forall\, x, y \in X\,;\, x \neq y$.

i.e., (X, τ) is T_0 – space iff the closure of singleton sets is deference if the elements are deference.

Proof : ($\Rightarrow$) Suppose that X is T_0 – space, to prove $\overline{\{x\}} \neq \overline{\{y\}}$ $\forall$ $x, y \in X\,;\, x \neq y$

$\because$ X is T_0 – space and $x \neq y \Rightarrow \exists\, U \in \tau\,;\, (x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)$

Suppose that $(x \in U \wedge y \notin U) \Rightarrow (x \in U \wedge y \in X - U)$

$X - U$ closed set since U open set $\Rightarrow \{y\} \subseteq X - U$

$\Rightarrow \overline{\{y\}} \subseteq \overline{X - U} = X - U$

(since $X - U$ closed and $\overline{X - U} = X - U$)

$\Rightarrow \overline{\{y\}} \subseteq X - U \;\wedge\; x \in U$

$\Rightarrow \{x\} \not\subseteq X - U$

$\Rightarrow \overline{\{x\}} \not\subseteq X - U$

$\therefore\ \overline{\{x\}} \neq \overline{\{y\}}$

By similar way if we take $(x \notin U \wedge y \in U)$

($\Leftarrow$) suppose that $\overline{\{x\}} \neq \overline{\{y\}}$ $\forall$ $x \neq y \in X$, to prove X is T_0 – space

Suppose that X is not T_0 – space $\Rightarrow (\exists\, x, y \in X\,;\, \forall\, U \in \tau\,;\, x \in U \Rightarrow y \in U)$ (def of T_0 – space) (i.e., every open set containing x its containing y)

Let $z \in X$; $z \in \overline{\{x\}}$ ----------(*)

$\Rightarrow \forall\, U \in \tau\,;\, z \in U \wedge U \bigcap \{x\} \neq \phi$

(since by true : $z \in \overline{A} \;\Leftrightarrow\; \forall\, U \in \tau\,;\, z \in U \;\wedge\; U \bigcap A \neq \phi$)

But, $U \bigcap \{x\} \neq \phi \;\Rightarrow\; x \in U$ (since the only element in $\{x\}$ is x)

$\therefore$ every set contains z must contains x. So, we have the following two statements : every open set contains z must contains x and every open set contains x must contains y.

$\therefore$ every open set contains z must contains y.

$\Rightarrow \forall\, U \in \tau\,;\, z \in U \wedge U \bigcap \{y\} \neq \phi$

$\Rightarrow z \in \overline{\{y\}}$ ----------(*)

$\Rightarrow \forall\, z \in \overline{\{x\}} \;\Rightarrow\; z \in \overline{\{y\}} \;\Rightarrow \overline{\{x\}} \subseteq \overline{\{y\}}$

By similar way we prove $\overline{\{y\}} \subseteq \overline{\{x\}}$

$\therefore$ $\overline{\{x\}} = \overline{\{y\}}$ C!! contribution (since $\overline{\{x\}} \neq \overline{\{y\}}$)

$\therefore$ X is T_0 – space.

Theorem : The property of being a T_0 – space is a hereditary property.

Proof :

Let (X, τ) T_0– space and (W, τ_W) subspace of X, to prove (W, τ_W) is T_0– space

Let $x, y \in W$; $x \neq y \Rightarrow x, y \in X$ (since $W \subseteq X$)

$\because$ X is T_0– space $\Rightarrow \exists\, U \in \tau$; $(x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)$

$\Rightarrow U \bigcap W \in \tau_W$ (by def. of τ_W)

; $(x \in U \bigcap W \wedge y \notin U \bigcap W) \vee (x \notin U \bigcap W \wedge y \in U \bigcap W)$

$\therefore$ (W, τ_W) is T_0– space.

Theorem : The property of being a T_0 – space is a topological property.

Proof :

Let $(X, \tau) \cong (Y, \tau')$ and suppose that X is T_0– space, to prove Y is T_0– space

$\because$ $(X, \tau) \cong (Y, \tau') \Rightarrow \exists\, f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f^{-1} continuous

Let $y_1, y_2 \in Y$; $y_1 \neq y_2 \Rightarrow f^{-1}(y_1), f^{-1}(y_2) \in X$

$\because$ f onto function $\Rightarrow f^{-1}(y_1) \neq \phi$, $f^{-1}(y_2) \neq \phi$

$\because$ f 1-1 function $\Rightarrow \exists!\, x_1 \in X$; $f^{-1}(y_1) = x_1$ and $\exists!\, x_2 \in X$; $f^{-1}(y_2) = x_2$

and $x_1 \neq x_2$ and $x_1, x_2 \in X$

$\because$ X is T_0– space $\Rightarrow \exists\, U \in \tau$; $(x_1 \in U \wedge x_2 \notin U) \vee (x_1 \notin U \wedge x_2 \in U)$

$\because$ f^{-1} is cont. or f open $\Rightarrow f(U) \in \tau'$; $(f(x_1) \in f(U) \wedge f(x_2) \notin f(U))$

$\vee$ $(f(x_1) \notin f(U) \wedge f(x_2) \in f(U))$

$\therefore$ Y is T_0– space

Similarly, we prove, if Y is T_0– space, then X is T_0– space.

Theorem : Let (X, τ) and (Y, τ') be two topological spaces. Then the product space $X \times Y$ is a T_0 – space iff each X and Y are T_0 – space.

Proof :

($\Rightarrow$) Suppose that $X \times Y$ is a T_0– space, to prove that X and Y are T_0– space

Let $x_1, x_2 \in X$; $x_1 \neq x_2$ and $y_1, y_2 \in Y$; $y_1 \neq y_2$

$\Rightarrow (x_1, y_1), (x_2, y_2) \in X \times Y$; $(x_1, y_1) \neq (x_2, y_2)$

$\because$ $X \times Y$ is a T_0– space $\Rightarrow \exists$ an open set $U \times V \in \tau_{X \times Y}$;

$((x_1, y_1) \in U \times V \wedge (x_2, y_2) \notin U \times V) \vee ((x_1, y_1) \notin U \times V \wedge (x_2, y_2) \in U \times V)$

$\Rightarrow \exists\, U \in \tau$; $(x_1 \in U \wedge x_2 \notin U) \vee (x_1 \notin U \wedge x_2 \in U) \Rightarrow$ X is a T_0– space

and $\exists\, V \in \tau'$; $(y_1 \in V \wedge y_2 \notin V) \vee (y_1 \notin V \wedge y_2 \in V) \Rightarrow$ Y is a T_0– space.

($\Leftarrow$) Suppose that X and Y are T_0 – space, to prove X×Y is a T_0 – space

Let $(x_1, y_1), (x_2, y_2) \in X \times Y$; $(x_1, y_1) \neq (x_2, y_2)$

By def. product space $\Rightarrow (x_1, x_2 \in X \wedge x_1 \neq x_2) \wedge (y_1, y_2 \in Y \wedge y_1 \neq y_2)$

$\because$ X is a T_0 – space $\Rightarrow \exists\, U \in \tau$; $(x_1 \in U \wedge x_2 \notin U) \vee (x_1 \notin U \wedge x_2 \in U)$

$\because$ Y is a T_0 – space $\Rightarrow \exists\, V \in \tau'$; $(y_1 \in V \wedge y_2 \notin V) \vee (y_1 \notin V \wedge y_2 \in V)$

$\Rightarrow \exists\, U \times V$ is a basic open set ; $((x_1, y_1) \in U \times V \wedge (x_2, y_2) \notin U \times V)$

$\vee ((x_1, y_1) \notin U \times V \wedge (x_2, y_2) \in U \times V)$

$\therefore$ X×Y is a T_0 – space.

Definition T_1 – Space:

Let (X, τ) be a topological space. Then the space (X, τ)is called **T_1 – space** iff for each pair of distinct points x, y ∈ X, there exists an open set in X containing x but not y, and an open set in X containing y but not x. i.e.,

X is T_1 – space $\Leftrightarrow \forall\, x, y \in X$; $x \neq y\ \exists\, U, V \in \tau$; $(x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V)$.

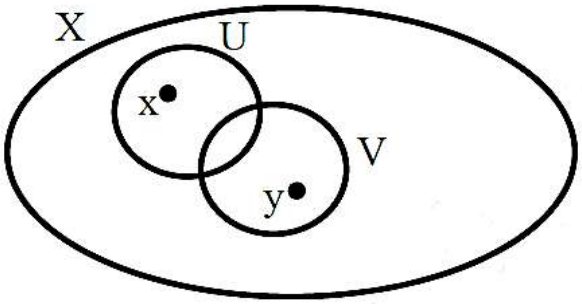
If (X, τ) is not T_1 – space, we define,

X is not T_1 – space $\Leftrightarrow \exists\, x, y \in X$; $x \neq y\ \forall\, U, V \in \tau$

; $(x \in U \wedge y \in U) \vee (x \notin U \wedge y \notin U)$

$(x \in V \wedge y \in V) \vee (x \notin V \wedge y \in V)$.

The following figure illustrate the definition of T_1 – space :



Remark : Every T_1 – space is T_0 – space (i.e., $T_1 \Rightarrow T_0$). But the reverse implications does not hold (i.e., $T_0 \not\Rightarrow T_1$) and the following example show that :

Example : Let (X, τ) be a topological space such that X = {1, 2, 3} and τ = {X, ϕ, {1}, {1, 2}}.

Solution : Clear (X, τ) is T_0 – space

But, (X, τ) is not T_1 – space, since 2 ≠ 3 and ∃ open set {1, 2} in X containing 2 but not 3, but ∄ open set in X containing 3 but not 2, since the only open set containing 3 is X and X containing 2 too.

Example : The cofinite topological space $(\mathbb{N}, \tau_{cof})$ is a T_1 – space.

Solution :

Let $n, m \in \mathbb{N}$; $n \neq m$, take $U = \mathbb{N} \setminus \{m\}$, $V = \mathbb{N} \setminus \{n\}$

$\Rightarrow U, V \in \tau_{cof}$ (since $U^c = (\mathbb{N} \setminus \{m\})^c = \{m\}$ finite set by def. of τ_{cof})

(since $V^c = (\mathbb{N} \setminus \{n\})^c = \{n\}$ finite set by def. of τ_{cof})

$\Rightarrow (n \in U = \mathbb{N} \setminus \{m\} \wedge m \notin U) \wedge (n \notin V = \mathbb{N} \setminus \{n\} \wedge m \in V)$

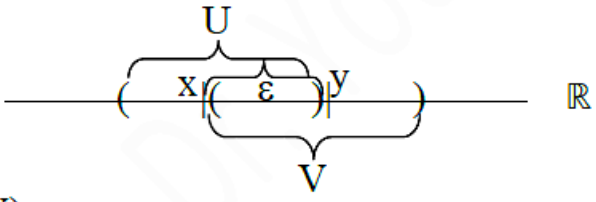
$\Rightarrow (\mathbb{N}, \tau_{cof})$ is T_1 – space.

Example : The usual topological space $(\mathbb{R}, \tau_u)$ is a T_1 – space.

Solution :

Let $x, y \in \mathbb{R}$; $x \neq y$, $\varepsilon = |x - y|$

Take $U = (x - \varepsilon , x + \varepsilon)$, $V = (y - \varepsilon, y + \varepsilon)$



$\therefore$ $U, V \in \tau_u$; $(x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V)$.

$\therefore$ $(\mathbb{R}, \tau_u)$ is T_1 – space.

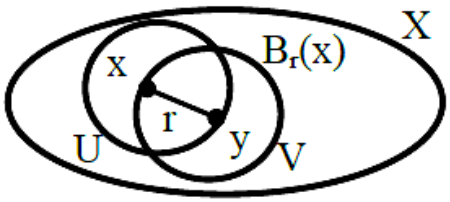
Theorem : Every metric space is a T_1 – space.

Proof : Let (X, d) be a metric space and $x, y \in X$; $x \neq y$

Take $U = B_r(x)$, $V = B_r(y)$; $r = d(x, y)$

$\therefore$ $U, V \in \tau_d$; $(x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V)$

$\therefore$ (X, d) is T_1 – space.



Theorem : The topological space (X, τ) is T_1 – space iff every singleton set $\{x\}$ is closed $\forall$ $x \in X$.

i.e., (X, τ) is T_1 – space iff every singleton set in X is closed.

Proof : ($\Rightarrow$) Suppose that X is T_1 – space, to prove $\{x\}$ closed $\forall$ $x \in X$

i.e., $X - \{x\}$ open set, we must prove $X - \{x\}$ contains a nbhd $\forall$ $y \in X - \{x\}$

Let $y \in X - \{x\} \Rightarrow x \neq y$

$\because$ X is T_1 – space $\Rightarrow \exists\, U, V_y \in \tau$; $(x \in U \wedge y \notin U) \wedge (x \notin V_y \wedge y \in V_y)$

$\Rightarrow y \in V_y \wedge x \notin V_y$

$\Rightarrow \{x\} \bigcap V_y = \phi$

$\Rightarrow V_y \subseteq X - \{x\} \wedge y \in V_y$

$\Rightarrow V_y \subseteq X - \{x\}$ $\forall$ $y \in X - \{x\}$

$\therefore$ X – {x} contains a nbhd $\forall$ $y \in X - \{x\}$.

$\therefore$ X – {x} open set $\Rightarrow$ {x} closed $\forall$ $x \in X$.

Corollary : If X is a T_1 – space, then every finite set is closed.

Proof : Let A be a finite set in X

$\Rightarrow$ $A = \{x_1, \dots . x_n\} = \bigcup_{i=1}^{n}\{x_i\}$

$\because$ (X, τ) is T_1 – space $\Rightarrow$ $\{x_i\} \in \mathcal{F}$ $\forall$ i

$\Rightarrow$ $\bigcup_{i=1}^{n}\{x_i\}$ closed

$\Rightarrow$ A is a closed

Corollary : If X is finite set and (X, τ) is a T_1 – space, then $\tau = D$.

Proof : To prove $\tau = D$ must we prove $(\forall x \in X \Rightarrow \{x\} \in \tau)$, i.e., every singleton set {x} is open.

Let $x \in X$

$\because$ X finite set $\Rightarrow$ X – {x} finite set

$\because$ X is T_1 – space $\Rightarrow$ X – {x} closed set

(by previous corollary : If X is T_1 – space, then every finite set is closed)

$\Rightarrow$ {x} open

$\therefore$ $\tau = D$.

Theorem : The property of being a T_1 – space is a hereditary property.

Proof :

Let (X, τ) T_1 – space and (W, τ_W) subspace of X, to prove (W, τ_W) T_1 – space

Let $x, y \in W$; $x \neq y$ $\Rightarrow$ $x, y \in X$ (since $W \subseteq X$)

$\because$ X is T_1 – space $\Rightarrow$ $\exists\, U, V \in \tau$; $(x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V)$.

$\Rightarrow$ $U \bigcap W \wedge V \bigcap W \in \tau_W$ (by def. τ_W)

$\Rightarrow$ $(x \in U \bigcap W \wedge y \notin U \bigcap W) \wedge (x \notin V \bigcap W \wedge y \in V \bigcap W)$.

$\therefore$ (W, τ_W) is a T_1 – space.

Theorem : The property of being a T_1 – space is a topological property.

Proof :

Let $(X, \tau) \cong (Y, \tau')$ and X is a T_1 – space, to prove Y is a T_1 – space

$\because$ $(X, \tau) \cong (Y, \tau')$ $\Rightarrow$ $\exists\, f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f^{-1} continuous

Let $y_1, y_2 \in Y$; $y_1 \neq y_2$ $\Rightarrow$ $f^{-1}(y_1), f^{-1}(y_2) \in X$

$\because$ f onto function $\Rightarrow f^{-1}(y_1) \neq \phi$, $f^{-1}(y_2) \neq \phi$

$\because$ f 1-1 function $\Rightarrow \exists!\ x_1 \in X\ ; f^{-1}(y_1) = x_1$ and $\exists!\ x_2 \in X\ ; f^{-1}(y_2) = x_2$ and $x_1 \neq x_2$ and $x_1, x_2 \in X$

$\because$ X is T_1 – space $\Rightarrow \exists\ U, V \in \tau\ ; (x_1 \in U \wedge x_2 \notin U) \wedge (x_1 \notin V \wedge x_2 \in V)$

$\because$ f^{-1} is cont. or f open $\Rightarrow f(U), f(V) \in \tau'\ ; (\ f(x_1) \in f(U) \wedge f(x_2) \notin f(U)\ )$ $\wedge\ (\ f(x_1) \notin f(V) \wedge f(x_2) \in f(V)\ )$

$\therefore$ Y is T_1 – space

By similar way we prove, if Y is T_1 – space, then X is T_1 – space.

Theorem (H.W.): Let (X, τ) and (Y, τ') be two topological spaces. Then the product space $X \times Y$ is a T_1 – space iff each X and Y are T_1 – space.

Definition : T_2 – Space or Hausdorff Space

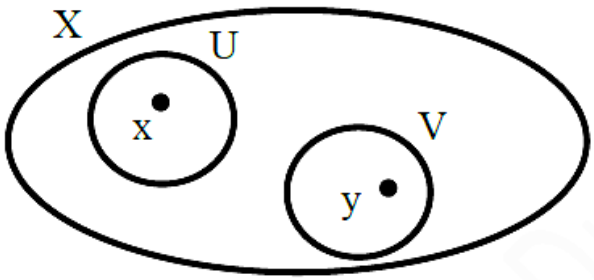
Let (X, τ) be a topological space. Then the space (X, τ) is called a **T_2 – space or Hausdorff space** iff for each pair of distinct points $x, y \in X$, there exist open sets U and V such that $x \in U$, $y \in V$, and $U \bigcap V = \phi$. i.e.,

X is T_2 – space $\Leftrightarrow \forall\ x, y \in X\ ; x \neq y\ \exists\ U, V \in \tau\ ; (x \in U \wedge y \in V), U \bigcap V = \phi$

If (X, τ) is not T_2 – space, we define,

X is not T_2–space $\Leftrightarrow \exists\ x, y \in X\ ; x \neq y\ \forall\ U, V \in \tau; U \bigcap V = \phi, (x, y \in U \vee x, y \in V)$

The following figure show the definition of T_2 – space :



Remark : Every T_2 – space is T_1 – space (i.e., $T_2 \Rightarrow T_1$). But the reverse implications do not hold (i.e., $T_1 \not\Rightarrow T_2$) and the following example show that :

Example : Take cofinite topology $(\mathbb{N}, \tau_{cof})$.

Solution : Clear $(\mathbb{N}, \tau_{cof})$ is T_1 – space (see page 81).

But, $(\mathbb{N}, \tau_{cof})$ is not T_2 – space, since if $n \neq m$, take $U = \mathbb{N} \setminus \{m\}$, $V = \mathbb{N} \setminus \{n\}$, but $U \bigcap V \neq \phi$. Therefore, $T_1 \not\Rightarrow T_2$.

Example : The space (X, D) is T_2 – space.

Solution :

Let $x, y \in X ; x \neq y \Rightarrow \{x\}, \{y\} \in D ; \{x\} \bigcap \{y\} = \phi , (x \in \{x\} \wedge y \in \{y\})$

$\Rightarrow$ (X, D) is T_2 – space.

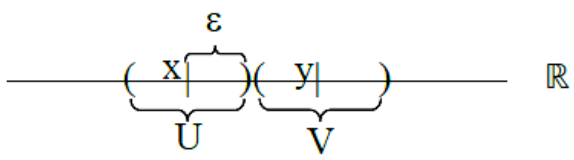
Example : The usual topological space $(\mathbb{R}, \tau_u)$ is T_2 – space.

Solution : Let $x, y \in \mathbb{R} \ ; x \neq y , \varepsilon = \frac{1}{2} | x - y |$

Take $U = (x - \varepsilon, x + \varepsilon), V = (y - \varepsilon, y + \varepsilon)$

$\therefore \ U, V \in \tau_u ; U \bigcap V = \phi , (x \in U \wedge y \in V)$

$\Rightarrow \ (\mathbb{R}, \tau_u)$ is T_2 – space.



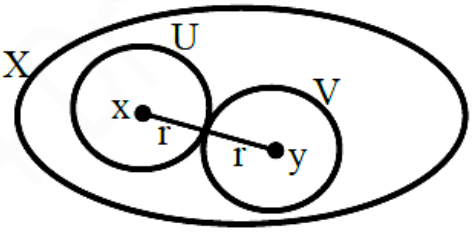
Theorem : Every metric space is T_2 – space.

Proof : Let (X, d) be a metric space and $x, y \in X \ ; x \neq y$

Take $U = B_r(x) , V = B_r(y) \ ; \ r = \frac{1}{2} \ d(x, y)$

$\therefore \ U, V \in \tau_d ; U \bigcap V = \phi , (x \in U \wedge y \in V)$

$\therefore$ (X, d) is T_2 – space.



Theorem : The property of being a T_2 – space is a hereditary property.

Proof :

Let (X, τ) T_2 – space and (W, τ_W) subspace of X, to prove (W, τ_W) T_2 – space

Let $x, y \in W ; x \neq y \Rightarrow x, y \in X$ (since $W \subseteq X$)

$\because$ X is T_2 – space $\Rightarrow \exists \ U, V \in \tau ; U \bigcap V = \phi , (x \in U \wedge y \in V)$.

$\Rightarrow U \bigcap W \wedge V \bigcap W \in \tau_W ,$ (by def. of τ_W)

$(U \bigcap W) \bigcap (U \bigcap W) = (U \bigcap V) \bigcap W = \phi \bigcap W = \phi$

and $(x \in U \bigcap W \wedge y \in V \bigcap W)$.

$\therefore \ (W, \tau_W)$ is T_2 – space.

Theorem : The property of being a T_2 – space is a topological property.

Proof :

Let $(X, \tau) \cong (Y, \tau')$ and suppose that Y is T_2 – space, to prove X is T_2 – space

$\because (X, \tau) \cong (Y, \tau') \Rightarrow \exists \ f : (X, \tau) \rightarrow (Y, \tau') ;$ f 1-1, f onto, f continuous, f^{-1} continuous

Let $x_1, x_2 \in X \ ; \ x_1 \neq x_2 \ \Rightarrow \ f(x_1), f(x_2) \in Y$

$\because$ f onto function $\Rightarrow f(x_1) \neq \phi, \ f(y_2) \neq \phi$

$\because$ f 1-1 function $\Rightarrow \exists!\ y_1 \in Y\ ;\ f(x_1) = y_1$ and $\exists!\ y_2 \in Y\ ;\ f(x_2) = y_2$
and $y_1 \neq y_2$, $y_1, y_2 \in Y$

$\because$ Y is T_2 – space $\Rightarrow \exists\ V_1, V_2 \in \tau'\ ;\ V_1 \bigcap V_2 = \phi\ ,\ (y_1 \in V_1 \wedge y_2 \in V_2)$

$\because$ f is continuous $\Rightarrow f^{-1}(V_1) = U_1, f^{-1}(V_2) = U_2 \in \tau\ ;$

$U_1 \bigcap U_2 = f^{-1}(V_1) \bigcap f^{-1}(V_2) = f^{-1}(V_1 \bigcap V_2) = f^{-1}(\phi) = \phi\ ,$
$(x_1 \in U_1 \wedge\ x_2 \in U_2)$

$\therefore$ X is T_2 – space

By similar way we prove, if X is T_2 – space, then Y is T_2 – space.

Theorem : Every compact set in T_2 – space is closed.

Proof : Let (X, τ) be T_2 – space and $A \subseteq X$; A compact in X

To prove A is closed in X, i.e., X – A open set in X

we must prove X – A contains an open nbhd $\forall\ x \in X - A$

(i.e., $\forall\ x \in X - A\ \exists\ U \in \tau\ ;\ x \in U \subseteq X - A$)

Let $x \in X - A \Rightarrow x \notin A \Rightarrow x \neq a\ \forall\ a \in A$

$\because$ X is T_2 – space $\Rightarrow\ \exists\ U_a, V_a \in \tau\ ;\ U_a \bigcap V_a = \phi\ ,\ (x \in U_a \wedge a \in Va)\ \forall\ a \in A.$

We have two family of open sets are $\{U_a\}_{a \in A}$ (every element in this family contains x) and $\{V_a\}_{a \in A}$ (every element in this family contains one of the elements A) and every U_a corresponding V_a such that $U_a \bigcap V_a = \phi$.

$\Rightarrow \{V_a\}_{a \in A}$ open cover of A, i.e., $A \subseteq \bigcup_{a \in A} V_a$

$\because$ A is compact set $\Rightarrow \exists\ a_1, \ldots, a_n\ ;\ A \subseteq \bigcup_{i=1}^{n} V_{a_i}$

Therefore, there is a finite family $\{U_a\}_{a \in A}$ corresponding the finite family $\{V_{a_i}\}_{i=1}^{n}$ which is $\{U_{a_i}\}_{i=1}^{n}$.

$\because$ every U_a contain x, then $x \in \bigcap_{i=1}^{n} U_{a_i}$

$\because$ every U_{a_i} is open set, then $\bigcap_{i=1}^{n} U_{a_i}$ is open set contain x

(second condition of def of top.)

Say, $U = \bigcap_{i=1}^{n} U_{a_i} \Rightarrow x \in U \in \tau$

On the other hand $\bigcup_{i=1}^{n} V_{a_i}$ is open set (third condition of def. of top.)

Say, $V = \bigcup_{i=1}^{n} V_{a_i} \Rightarrow A \subseteq V \in \tau$

Notes that,

$U \bigcap V = \phi$ (since $U_a \bigcap V_a = \phi$)

$\Rightarrow A \bigcap U = \phi$ (since $A \subseteq V$ and $U \bigcap V = \phi$)

$\Rightarrow U \subseteq X - A$

$\Rightarrow x \in U \subseteq X - A \wedge U \in \tau$

$\Rightarrow X - A$ open set in $X \; \forall \; x \in X - A$

$\Rightarrow$ A closed set in X.

Theorem : If $f : (X, \tau) \to (Y, \tau')$ is continuous bijective function and X is compact space and Y is T_2 – space, the f is homomorphism.

Proof : It is enough to prove f is closed function

Let F be a closed set in X

$\because$ X is compact and F closed in $X \Rightarrow$ F compact in X

(by theorem : A closed subset of compact space is compact)

$\because$ f is continuous and F is compact in $X \Rightarrow f(F)$ is compact in Y

(by theorem : A continuous image of compact set is compact set)

$\because$ Y is T_2 – space and f(F) compact in $Y \Rightarrow f(F)$ is closed in Y

(by previous theorem : Every compact set in T_2 – space is closed)

$\therefore$ f is closed function $\Rightarrow$ f is Homeomorphism.

Theorem : If A is closed set and B is compact set in T_2 – space (X, τ), then $A \bigcap B$ is compact.

Proof :

$\because$ X is T_2 – space and B compact in $X \Rightarrow$ B is closed in X

(by theorem : Every compact set in T_2 – space is closed)

$\because$ A and B closed sets $\Rightarrow A \bigcap B$ closed set i.e., $A \bigcap B \in \mathcal{F}$

(second condition of def of top.)

$\because$ $A \bigcap B \subseteq B$ i.e., $A \bigcap B$ subspace of B and

$A \bigcap B$ closed in B and B compact $\Rightarrow$ $A \bigcap B$ compact

(by theorem : A closed subset of compact space is compact)

Separation Axioms

Many properties of topological space X depend upon the distribution of the open sets in the space. Roughly speaking, a space is more likely to be separable, or first or second countable, if there are few open sets; in the other hand, an arbitrary function on X to some topological space is more likely to be continuous, or sequence to have a unique limit, if the space has many open sets.

The separation axioms of Alexandroff and Hopf, discussed in this section, postulate or discuss the existence of enough open sets.

Definition T_0 – Space:

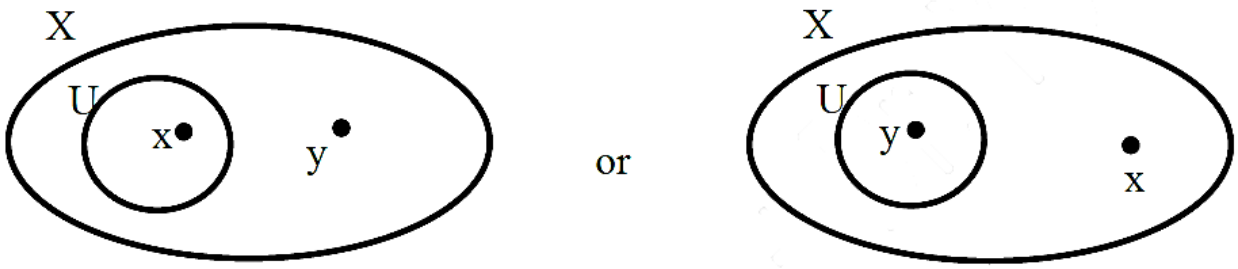
Let (X, τ) be a topological space. Then the space $(X, \tau)$is called **T_0 – space** iff for each pair of distinct points $x, y \in X$, there is either an open set containing x but not y or an open set containing y but not x. i.e.,

X is T_0 – space $\Leftrightarrow \forall\, x, y \in X \;;\; x \neq y; \exists\, U \in \tau \;; (x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)$.

If (X, τ) is not T_0 – space, we define,

X is not T_0 – space $\Leftrightarrow \exists\, x, y \in X \;;\; x \neq y \;\; \forall\, U \in \tau \;; (x, y \in U) \vee (x, y \notin U)$.

The following figure show the definition of T_0 – Space :



Example : Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1\}, \{1, 2\}\}$. Is (X, τ), T_0 – space ?

Solution: We must test the definitionfor and two points in X satisfies the definition or not

$1 \neq 2 \Rightarrow \exists$ open set $\{1\} \in \tau \;;\; 1 \in \{1\} \wedge 2 \notin \{1\}$

$1 \neq 3 \Rightarrow \exists$ open set $\{1\} \in \tau \;;\; 1 \in \{1\} \wedge 3 \notin \{1\}$

$2 \neq 3 \Rightarrow \exists$ open set $\{1, 2\} \in \tau \;;\; 2 \in \{1, 2\} \wedge 3 \notin \{1, 2\}$

$\therefore$ The definition is satisfied $\Rightarrow (X, \tau)$ is T_0 – space.

Example : Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$. Is (X, τ), T_0 – space ?

Solution : Since $b \neq c \Rightarrow \nexists$ open set containing b but not c or an open set containing c but not b.

Clear that X containing b, c and ϕ open set not containing b, c and $\{a\}$ open set not containing b, c.

Therefore, (X, τ) is not T_0 – space.

Example : Is $(\mathbb{N}, \tau_{cof})$ T_0 – space.

Solution : Since in general since $\mathbb{N}$ containing infinite numbers, then:

Let $n, m \in \mathbb{N}$; $n \neq m$, to prove $\exists$ open set $U \in \tau_{cof}$; $m \in U \wedge n \notin U$ or vice versa.

Take $U = \mathbb{N} \setminus \{n\} \Rightarrow m \in U \wedge n \notin U$

and $U \in \tau_{cof}$ (since $U^c = (\mathbb{N} \setminus \{n\})^c = \{n\}$ finite set by def. of τ_{cof})

Similarly, way we can take : $U = \mathbb{N} \setminus \{m\} \Rightarrow m \notin U \wedge n \in U$

and $U \in \tau_{cof}$ (since $U^c = (\mathbb{N} \setminus \{m\})^c = \{m\}$ finite set by def. of τ_{cof})

The two cases similar and satisfy the definition $\Rightarrow (\mathbb{N}, \tau_{cof})$ is T_0 – space.

Example : In the space (X, I) if X is any set containing more than one element, then (X, I) is not T_0 – space, since X contains more than one element we take $x, y \in X$; $x \neq y$ and $\nexists$ open set containing x but not y or an open set containing y but not x and ϕ open set not containing x, y.

Example : The space (X, D) is T_0 – space.

Solution : Let $x, y \in X$; $x \neq y$, then $\{x\} \in D$ i.e., $\{x\}$ open set (by definition D since $D = IP(X)$), hence $x \in \{x\}$ and $y \notin \{x\}$. We can take $\{y\}$ replace of $\{x\}$ and $y \in \{y\}$ and $x \notin \{y\}$. Therefore, (X, D) is T_0 – space.

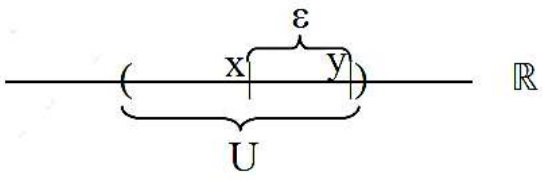
Example : The usual topological space $(\mathbb{R}, \tau_u)$ is T_0 – space.

Solution : Let $x, y \in \mathbb{R}$; $x \neq y$

Take $U = (x - \varepsilon, x + \varepsilon)$; $\varepsilon = |x - y|$

$\Rightarrow U \in \tau_u \wedge x \in U \wedge y \notin U$

$\therefore$ $(\mathbb{R}, \tau_u)$ is T_0 – space.



Theorem : (X, τ) is T_0 – space iff $\overline{\{x\}} \neq \overline{\{y\}}$ $\forall\, x, y \in X\,;\, x \neq y$.

i.e., (X, τ) is T_0 – space iff the closure of singleton sets is deference if the elements are deference.

Proof : ($\Rightarrow$) Suppose that X is T_0 – space, to prove $\overline{\{x\}} \neq \overline{\{y\}}$ $\forall$ $x, y \in X\,;\, x \neq y$

$\because$ X is T_0 – space and $x \neq y \Rightarrow \exists\, U \in \tau\,;\, (x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)$

Suppose that $(x \in U \wedge y \notin U) \Rightarrow (x \in U \wedge y \in X - U)$

$X - U$ closed set since U open set $\Rightarrow \{y\} \subseteq X - U$

$\Rightarrow \overline{\{y\}} \subseteq \overline{X - U} = X - U$

(since $X - U$ closed and $\overline{X - U} = X - U$)

$\Rightarrow \overline{\{y\}} \subseteq X - U \;\wedge\; x \in U$

$\Rightarrow \{x\} \not\subseteq X - U$

$\Rightarrow \overline{\{x\}} \not\subseteq X - U$

$\therefore\ \overline{\{x\}} \neq \overline{\{y\}}$

By similar way if we take $(x \notin U \wedge y \in U)$

($\Leftarrow$) suppose that $\overline{\{x\}} \neq \overline{\{y\}}$ $\forall$ $x \neq y \in X$, to prove X is T_0 – space

Suppose that X is not T_0 – space $\Rightarrow (\exists\, x, y \in X\,;\, \forall\, U \in \tau\,;\, x \in U \Rightarrow y \in U)$ (def of T_0 – space) (i.e., every open set containing x its containing y)

Let $z \in X$; $z \in \overline{\{x\}}$ ----------(*)

$\Rightarrow \forall\, U \in \tau\,;\, z \in U \wedge U \bigcap \{x\} \neq \phi$

(since by true : $z \in \overline{A} \;\Leftrightarrow\; \forall\, U \in \tau\,;\, z \in U \;\wedge\; U \bigcap A \neq \phi$)

But, $U \bigcap \{x\} \neq \phi \;\Rightarrow\; x \in U$ (since the only element in $\{x\}$ is x)

$\therefore$ every set contains z must contains x. So, we have the following two statements : every open set contains z must contains x and every open set contains x must contains y.

$\therefore$ every open set contains z must contains y.

$\Rightarrow \forall\, U \in \tau\,;\, z \in U \wedge U \bigcap \{y\} \neq \phi$

$\Rightarrow z \in \overline{\{y\}}$ ----------(*)

$\Rightarrow \forall\, z \in \overline{\{x\}} \;\Rightarrow\; z \in \overline{\{y\}} \;\Rightarrow \overline{\{x\}} \subseteq \overline{\{y\}}$

By similar way we prove $\overline{\{y\}} \subseteq \overline{\{x\}}$

$\therefore$ $\overline{\{x\}} = \overline{\{y\}}$ C!! contribution (since $\overline{\{x\}} \neq \overline{\{y\}}$)

$\therefore$ X is T_0 – space.

Theorem : The property of being a T_0 – space is a hereditary property.

Proof :

Let (X, τ) T_0– space and (W, τ_W) subspace of X, to prove (W, τ_W) is T_0– space

Let $x, y \in W$; $x \neq y \Rightarrow x, y \in X$ (since $W \subseteq X$)

$\because$ X is T_0– space $\Rightarrow \exists\, U \in \tau$; $(x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)$

$\Rightarrow U \bigcap W \in \tau_W$ (by def. of τ_W)

; $(x \in U \bigcap W \wedge y \notin U \bigcap W) \vee (x \notin U \bigcap W \wedge y \in U \bigcap W)$

$\therefore$ (W, τ_W) is T_0– space.

Theorem : The property of being a T_0 – space is a topological property.

Proof :

Let $(X, \tau) \cong (Y, \tau')$ and suppose that X is T_0– space, to prove Y is T_0– space

$\because$ $(X, \tau) \cong (Y, \tau') \Rightarrow \exists\, f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f^{-1} continuous

Let $y_1, y_2 \in Y$; $y_1 \neq y_2 \Rightarrow f^{-1}(y_1), f^{-1}(y_2) \in X$

$\because$ f onto function $\Rightarrow f^{-1}(y_1) \neq \phi$, $f^{-1}(y_2) \neq \phi$

$\because$ f 1-1 function $\Rightarrow \exists!\, x_1 \in X$; $f^{-1}(y_1) = x_1$ and $\exists!\, x_2 \in X$; $f^{-1}(y_2) = x_2$

and $x_1 \neq x_2$ and $x_1, x_2 \in X$

$\because$ X is T_0– space $\Rightarrow \exists\, U \in \tau$; $(x_1 \in U \wedge x_2 \notin U) \vee (x_1 \notin U \wedge x_2 \in U)$

$\because$ f^{-1} is cont. or f open $\Rightarrow f(U) \in \tau'$; $(f(x_1) \in f(U) \wedge f(x_2) \notin f(U))$

$\vee$ $(f(x_1) \notin f(U) \wedge f(x_2) \in f(U))$

$\therefore$ Y is T_0– space

Similarly, we prove, if Y is T_0– space, then X is T_0– space.

Theorem : Let (X, τ) and (Y, τ') be two topological spaces. Then the product space $X \times Y$ is a T_0 – space iff each X and Y are T_0 – space.

Proof :

($\Rightarrow$) Suppose that $X \times Y$ is a T_0– space, to prove that X and Y are T_0– space

Let $x_1, x_2 \in X$; $x_1 \neq x_2$ and $y_1, y_2 \in Y$; $y_1 \neq y_2$

$\Rightarrow (x_1, y_1), (x_2, y_2) \in X \times Y$; $(x_1, y_1) \neq (x_2, y_2)$

$\because$ $X \times Y$ is a T_0– space $\Rightarrow \exists$ an open set $U \times V \in \tau_{X \times Y}$;

$((x_1, y_1) \in U \times V \wedge (x_2, y_2) \notin U \times V) \vee ((x_1, y_1) \notin U \times V \wedge (x_2, y_2) \in U \times V)$

$\Rightarrow \exists\, U \in \tau$; $(x_1 \in U \wedge x_2 \notin U) \vee (x_1 \notin U \wedge x_2 \in U) \Rightarrow$ X is a T_0– space

and $\exists\, V \in \tau'$; $(y_1 \in V \wedge y_2 \notin V) \vee (y_1 \notin V \wedge y_2 \in V) \Rightarrow$ Y is a T_0– space.

($\Leftarrow$) Suppose that X and Y are T_0 – space, to prove X×Y is a T_0 – space

Let $(x_1, y_1), (x_2, y_2) \in X \times Y$; $(x_1, y_1) \neq (x_2, y_2)$

By def. product space $\Rightarrow (x_1, x_2 \in X \wedge x_1 \neq x_2) \wedge (y_1, y_2 \in Y \wedge y_1 \neq y_2)$

$\because$ X is a T_0 – space $\Rightarrow \exists\, U \in \tau$; $(x_1 \in U \wedge x_2 \notin U) \vee (x_1 \notin U \wedge x_2 \in U)$

$\because$ Y is a T_0 – space $\Rightarrow \exists\, V \in \tau'$; $(y_1 \in V \wedge y_2 \notin V) \vee (y_1 \notin V \wedge y_2 \in V)$

$\Rightarrow \exists\, U \times V$ is a basic open set ; $((x_1, y_1) \in U \times V \wedge (x_2, y_2) \notin U \times V)$

$\vee ((x_1, y_1) \notin U \times V \wedge (x_2, y_2) \in U \times V)$

$\therefore$ X×Y is a T_0 – space.

Definition T_1 – Space:

Let (X, τ) be a topological space. Then the space (X, τ)is called **T_1 – space** iff for each pair of distinct points x, y ∈ X, there exists an open set in X containing x but not y, and an open set in X containing y but not x. i.e.,

X is T_1 – space $\Leftrightarrow \forall\, x, y \in X$; $x \neq y\ \exists\, U, V \in \tau$; $(x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V)$.

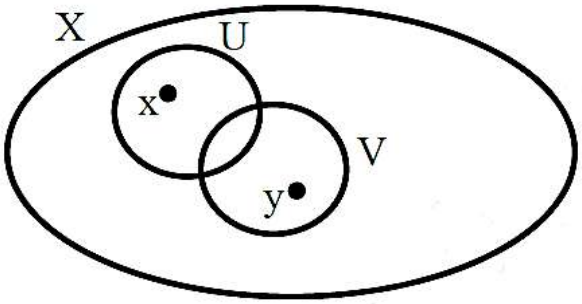
If (X, τ) is not T_1 – space, we define,

X is not T_1 – space $\Leftrightarrow \exists\, x, y \in X$; $x \neq y\ \forall\, U, V \in \tau$

; $(x \in U \wedge y \in U) \vee (x \notin U \wedge y \notin U)$

$(x \in V \wedge y \in V) \vee (x \notin V \wedge y \in V)$.

The following figure illustrate the definition of T_1 – space :



Remark : Every T_1 – space is T_0 – space (i.e., $T_1 \Rightarrow T_0$). But the reverse implications does not hold (i.e., $T_0 \not\Rightarrow T_1$) and the following example show that :

Example : Let (X, τ) be a topological space such that X = {1, 2, 3} and τ = {X, ϕ, {1}, {1, 2}}.

Solution : Clear (X, τ) is T_0 – space

But, (X, τ) is not T_1 – space, since 2 ≠ 3 and ∃ open set {1, 2} in X containing 2 but not 3, but ∄ open set in X containing 3 but not 2, since the only open set containing 3 is X and X containing 2 too.

Example : The cofinite topological space $(\mathbb{N}, \tau_{cof})$ is a T_1 – space.

Solution :

Let $n, m \in \mathbb{N}$; $n \neq m$, take $U = \mathbb{N} \setminus \{m\}$, $V = \mathbb{N} \setminus \{n\}$

$\Rightarrow U, V \in \tau_{cof}$ (since $U^c = (\mathbb{N} \setminus \{m\})^c = \{m\}$ finite set by def. of τ_{cof})

(since $V^c = (\mathbb{N} \setminus \{n\})^c = \{n\}$ finite set by def. of τ_{cof})

$\Rightarrow (n \in U = \mathbb{N} \setminus \{m\} \wedge m \notin U) \wedge (n \notin V = \mathbb{N} \setminus \{n\} \wedge m \in V)$

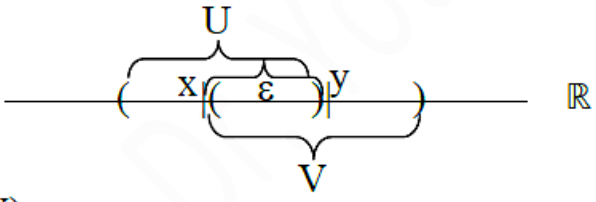
$\Rightarrow (\mathbb{N}, \tau_{cof})$ is T_1 – space.

Example : The usual topological space $(\mathbb{R}, \tau_u)$ is a T_1 – space.

Solution :

Let $x, y \in \mathbb{R}$; $x \neq y$, $\varepsilon = |x - y|$

Take $U = (x - \varepsilon , x + \varepsilon)$, $V = (y - \varepsilon, y + \varepsilon)$



$\therefore$ $U, V \in \tau_u$; $(x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V)$.

$\therefore$ $(\mathbb{R}, \tau_u)$ is T_1 – space.

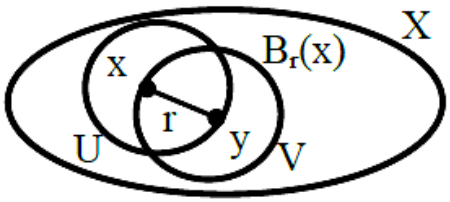
Theorem : Every metric space is a T_1 – space.

Proof : Let (X, d) be a metric space and $x, y \in X$; $x \neq y$

Take $U = B_r(x)$, $V = B_r(y)$; $r = d(x, y)$

$\therefore$ $U, V \in \tau_d$; $(x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V)$

$\therefore$ (X, d) is T_1 – space.



Theorem : The topological space (X, τ) is T_1 – space iff every singleton set $\{x\}$ is closed $\forall$ $x \in X$.

i.e., (X, τ) is T_1 – space iff every singleton set in X is closed.

Proof : ($\Rightarrow$) Suppose that X is T_1 – space, to prove $\{x\}$ closed $\forall$ $x \in X$

i.e., $X - \{x\}$ open set, we must prove $X - \{x\}$ contains a nbhd $\forall$ $y \in X - \{x\}$

Let $y \in X - \{x\} \Rightarrow x \neq y$

$\because$ X is T_1 – space $\Rightarrow \exists\, U, V_y \in \tau$; $(x \in U \wedge y \notin U) \wedge (x \notin V_y \wedge y \in V_y)$

$\Rightarrow y \in V_y \wedge x \notin V_y$

$\Rightarrow \{x\} \bigcap V_y = \phi$

$\Rightarrow V_y \subseteq X - \{x\} \wedge y \in V_y$

$\Rightarrow V_y \subseteq X - \{x\}$ $\forall$ $y \in X - \{x\}$

$\therefore$ X – {x} contains a nbhd $\forall$ $y \in X - \{x\}$.

$\therefore$ X – {x} open set $\Rightarrow$ {x} closed $\forall$ $x \in X$.

Corollary : If X is a T_1 – space, then every finite set is closed.

Proof : Let A be a finite set in X

$\Rightarrow$ $A = \{x_1, \dots . x_n\} = \bigcup_{i=1}^{n}\{x_i\}$

$\because$ (X, τ) is T_1 – space $\Rightarrow$ $\{x_i\} \in \mathcal{F}$ $\forall$ i

$\Rightarrow$ $\bigcup_{i=1}^{n}\{x_i\}$ closed

$\Rightarrow$ A is a closed

Corollary : If X is finite set and (X, τ) is a T_1 – space, then $\tau = D$.

Proof : To prove $\tau = D$ must we prove $(\forall x \in X \Rightarrow \{x\} \in \tau)$, i.e., every singleton set {x} is open.

Let $x \in X$

$\because$ X finite set $\Rightarrow$ X – {x} finite set

$\because$ X is T_1 – space $\Rightarrow$ X – {x} closed set

(by previous corollary : If X is T_1 – space, then every finite set is closed)

$\Rightarrow$ {x} open

$\therefore$ $\tau = D$.

Theorem : The property of being a T_1 – space is a hereditary property.

Proof :

Let (X, τ) T_1 – space and (W, τ_W) subspace of X, to prove (W, τ_W) T_1 – space

Let $x, y \in W$; $x \neq y$ $\Rightarrow$ $x, y \in X$ (since $W \subseteq X$)

$\because$ X is T_1 – space $\Rightarrow$ $\exists\, U, V \in \tau$; $(x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V)$.

$\Rightarrow$ $U \bigcap W \wedge V \bigcap W \in \tau_W$ (by def. τ_W)

$\Rightarrow$ $(x \in U \bigcap W \wedge y \notin U \bigcap W) \wedge (x \notin V \bigcap W \wedge y \in V \bigcap W)$.

$\therefore$ (W, τ_W) is a T_1 – space.

Theorem : The property of being a T_1 – space is a topological property.

Proof :

Let $(X, \tau) \cong (Y, \tau')$ and X is a T_1 – space, to prove Y is a T_1 – space

$\because$ $(X, \tau) \cong (Y, \tau')$ $\Rightarrow$ $\exists\, f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f^{-1} continuous

Let $y_1, y_2 \in Y$; $y_1 \neq y_2$ $\Rightarrow$ $f^{-1}(y_1), f^{-1}(y_2) \in X$

$\because$ f onto function $\Rightarrow f^{-1}(y_1) \neq \phi$, $f^{-1}(y_2) \neq \phi$

$\because$ f 1-1 function $\Rightarrow \exists!\ x_1 \in X\ ; f^{-1}(y_1) = x_1$ and $\exists!\ x_2 \in X\ ; f^{-1}(y_2) = x_2$ and $x_1 \neq x_2$ and $x_1, x_2 \in X$

$\because$ X is T_1 – space $\Rightarrow \exists\ U, V \in \tau\ ; (x_1 \in U \wedge x_2 \notin U) \wedge (x_1 \notin V \wedge x_2 \in V)$

$\because$ f^{-1} is cont. or f open $\Rightarrow f(U), f(V) \in \tau'\ ; (\ f(x_1) \in f(U) \wedge f(x_2) \notin f(U)\ )$ $\wedge\ (\ f(x_1) \notin f(V) \wedge f(x_2) \in f(V)\ )$

$\therefore$ Y is T_1 – space

By similar way we prove, if Y is T_1 – space, then X is T_1 – space.

Theorem (H.W.): Let (X, τ) and (Y, τ') be two topological spaces. Then the product space $X \times Y$ is a T_1 – space iff each X and Y are T_1 – space.

Definition : T_2 – Space or Hausdorff Space

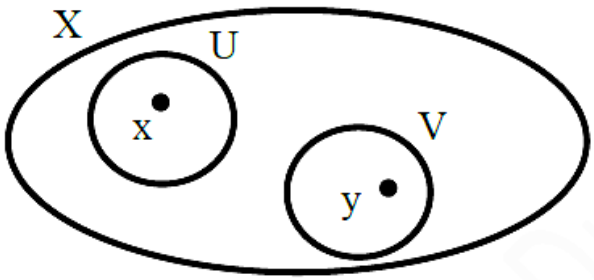
Let (X, τ) be a topological space. Then the space (X, τ) is called a **T_2 – space or Hausdorff space** iff for each pair of distinct points $x, y \in X$, there exist open sets U and V such that $x \in U$, $y \in V$, and $U \bigcap V = \phi$. i.e.,

X is T_2 – space $\Leftrightarrow \forall\ x, y \in X\ ; x \neq y\ \exists\ U, V \in \tau\ ; (x \in U \wedge y \in V), U \bigcap V = \phi$

If (X, τ) is not T_2 – space, we define,

X is not T_2–space $\Leftrightarrow \exists\ x, y \in X\ ; x \neq y\ \forall\ U, V \in \tau; U \bigcap V = \phi, (x, y \in U \vee x, y \in V)$

The following figure show the definition of T_2 – space :



Remark : Every T_2 – space is T_1 – space (i.e., $T_2 \Rightarrow T_1$). But the reverse implications do not hold (i.e., $T_1 \not\Rightarrow T_2$) and the following example show that :

Example : Take cofinite topology $(\mathbb{N}, \tau_{cof})$.

Solution : Clear $(\mathbb{N}, \tau_{cof})$ is T_1 – space (see page 81).

But, $(\mathbb{N}, \tau_{cof})$ is not T_2 – space, since if $n \neq m$, take $U = \mathbb{N} \setminus \{m\}$, $V = \mathbb{N} \setminus \{n\}$, but $U \bigcap V \neq \phi$. Therefore, $T_1 \not\Rightarrow T_2$.

Example : The space (X, D) is T_2 – space.

Solution :

Let $x, y \in X ; x \neq y \Rightarrow \{x\}, \{y\} \in D ; \{x\} \bigcap \{y\} = \phi , (x \in \{x\} \wedge y \in \{y\})$

$\Rightarrow$ (X, D) is T_2 – space.

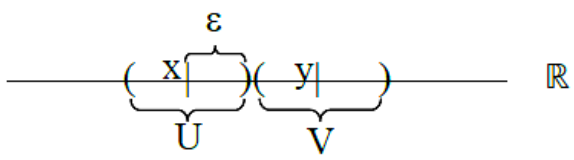
Example : The usual topological space $(\mathbb{R}, \tau_u)$ is T_2 – space.

Solution : Let $x, y \in \mathbb{R} \ ; x \neq y , \varepsilon = \frac{1}{2} | x - y |$

Take $U = (x - \varepsilon, x + \varepsilon), V = (y - \varepsilon, y + \varepsilon)$

$\therefore \ U, V \in \tau_u ; U \bigcap V = \phi , (x \in U \wedge y \in V)$

$\Rightarrow \ (\mathbb{R}, \tau_u)$ is T_2 – space.



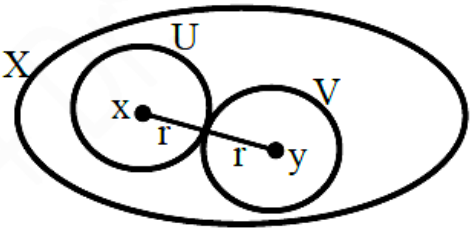
Theorem : Every metric space is T_2 – space.

Proof : Let (X, d) be a metric space and $x, y \in X \ ; x \neq y$

Take $U = B_r(x) , V = B_r(y) \ ; \ r = \frac{1}{2} \ d(x, y)$

$\therefore \ U, V \in \tau_d ; U \bigcap V = \phi , (x \in U \wedge y \in V)$

$\therefore$ (X, d) is T_2 – space.



Theorem : The property of being a T_2 – space is a hereditary property.

Proof :

Let (X, τ) T_2 – space and (W, τ_W) subspace of X, to prove (W, τ_W) T_2 – space

Let $x, y \in W ; x \neq y \Rightarrow x, y \in X$ (since $W \subseteq X$)

$\because$ X is T_2 – space $\Rightarrow \exists \ U, V \in \tau ; U \bigcap V = \phi , (x \in U \wedge y \in V)$.

$\Rightarrow U \bigcap W \wedge V \bigcap W \in \tau_W ,$ (by def. of τ_W)

$(U \bigcap W) \bigcap (U \bigcap W) = (U \bigcap V) \bigcap W = \phi \bigcap W = \phi$

and $(x \in U \bigcap W \wedge y \in V \bigcap W)$.

$\therefore \ (W, \tau_W)$ is T_2 – space.

Theorem : The property of being a T_2 – space is a topological property.

Proof :

Let $(X, \tau) \cong (Y, \tau')$ and suppose that Y is T_2 – space, to prove X is T_2 – space

$\because (X, \tau) \cong (Y, \tau') \Rightarrow \exists \ f : (X, \tau) \rightarrow (Y, \tau') ;$ f 1-1, f onto, f continuous, f^{-1} continuous

Let $x_1, x_2 \in X \ ; \ x_1 \neq x_2 \ \Rightarrow \ f(x_1), f(x_2) \in Y$

$\because$ f onto function $\Rightarrow f(x_1) \neq \phi, \ f(y_2) \neq \phi$

$\because$ f 1-1 function $\Rightarrow \exists!\ y_1 \in Y\ ;\ f(x_1) = y_1$ and $\exists!\ y_2 \in Y\ ;\ f(x_2) = y_2$
and $y_1 \neq y_2$, $y_1, y_2 \in Y$

$\because$ Y is T_2 – space $\Rightarrow \exists\ V_1, V_2 \in \tau'\ ;\ V_1 \bigcap V_2 = \phi\ ,\ (y_1 \in V_1 \wedge y_2 \in V_2)$

$\because$ f is continuous $\Rightarrow f^{-1}(V_1) = U_1, f^{-1}(V_2) = U_2 \in \tau\ ;$

$U_1 \bigcap U_2 = f^{-1}(V_1) \bigcap f^{-1}(V_2) = f^{-1}(V_1 \bigcap V_2) = f^{-1}(\phi) = \phi\ ,$
$(x_1 \in U_1 \wedge\ x_2 \in U_2)$

$\therefore$ X is T_2 – space

By similar way we prove, if X is T_2 – space, then Y is T_2 – space.

Theorem : Every compact set in T_2 – space is closed.

Proof : Let (X, τ) be T_2 – space and $A \subseteq X$; A compact in X

To prove A is closed in X, i.e., X – A open set in X

we must prove X – A contains an open nbhd $\forall\ x \in X - A$

(i.e., $\forall\ x \in X - A\ \exists\ U \in \tau\ ;\ x \in U \subseteq X - A$)

Let $x \in X - A \Rightarrow x \notin A \Rightarrow x \neq a\ \forall\ a \in A$

$\because$ X is T_2 – space $\Rightarrow\ \exists\ U_a, V_a \in \tau\ ;\ U_a \bigcap V_a = \phi\ ,\ (x \in U_a \wedge a \in Va)\ \forall\ a \in A.$

We have two family of open sets are $\{U_a\}_{a \in A}$ (every element in this family contains x) and $\{V_a\}_{a \in A}$ (every element in this family contains one of the elements A) and every U_a corresponding V_a such that $U_a \bigcap V_a = \phi$.

$\Rightarrow \{V_a\}_{a \in A}$ open cover of A, i.e., $A \subseteq \bigcup_{a \in A} V_a$

$\because$ A is compact set $\Rightarrow \exists\ a_1, \ldots, a_n\ ;\ A \subseteq \bigcup_{i=1}^{n} V_{a_i}$

Therefore, there is a finite family $\{U_a\}_{a \in A}$ corresponding the finite family $\{V_{a_i}\}_{i=1}^{n}$ which is $\{U_{a_i}\}_{i=1}^{n}$.

$\because$ every U_a contain x, then $x \in \bigcap_{i=1}^{n} U_{a_i}$

$\because$ every U_{a_i} is open set, then $\bigcap_{i=1}^{n} U_{a_i}$ is open set contain x

(second condition of def of top.)

Say, $U = \bigcap_{i=1}^{n} U_{a_i} \Rightarrow x \in U \in \tau$

On the other hand $\bigcup_{i=1}^{n} V_{a_i}$ is open set (third condition of def. of top.)

Say, $V = \bigcup_{i=1}^{n} V_{a_i} \Rightarrow A \subseteq V \in \tau$

Notes that,

$U \bigcap V = \phi$ (since $U_a \bigcap V_a = \phi$)

$\Rightarrow A \bigcap U = \phi$ (since $A \subseteq V$ and $U \bigcap V = \phi$)

$\Rightarrow U \subseteq X - A$

$\Rightarrow x \in U \subseteq X - A \wedge U \in \tau$

$\Rightarrow X - A$ open set in $X \; \forall \; x \in X - A$

$\Rightarrow$ A closed set in X.

Theorem : If $f : (X, \tau) \to (Y, \tau')$ is continuous bijective function and X is compact space and Y is T_2 – space, the f is homomorphism.

Proof : It is enough to prove f is closed function

Let F be a closed set in X

$\because$ X is compact and F closed in $X \Rightarrow$ F compact in X

(by theorem : A closed subset of compact space is compact)

$\because$ f is continuous and F is compact in $X \Rightarrow f(F)$ is compact in Y

(by theorem : A continuous image of compact set is compact set)

$\because$ Y is T_2 – space and f(F) compact in $Y \Rightarrow f(F)$ is closed in Y

(by previous theorem : Every compact set in T_2 – space is closed)

$\therefore$ f is closed function $\Rightarrow$ f is Homeomorphism.

Theorem : If A is closed set and B is compact set in T_2 – space (X, τ), then $A \bigcap B$ is compact.

Proof :

$\because$ X is T_2 – space and B compact in $X \Rightarrow$ B is closed in X

(by theorem : Every compact set in T_2 – space is closed)

$\because$ A and B closed sets $\Rightarrow A \bigcap B$ closed set i.e., $A \bigcap B \in \mathcal{F}$

(second condition of def of top.)

$\because$ $A \bigcap B \subseteq B$ i.e., $A \bigcap B$ subspace of B and

$A \bigcap B$ closed in B and B compact $\Rightarrow$ $A \bigcap B$ compact

(by theorem : A closed subset of compact space is compact)

Example (6) :

Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{a\}, \{a, b\}\}$

Define the function $f : (X, \tau) \to (Y, \tau')$; $f(1) = f(2) = a$, $f(3) = b$

Show that f is not continuous, open, not closed.

Example (7) :

Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{c\}, \{b, c\}\}$

Define the function $f : (X, \tau) \to (Y, \tau')$; $f(1) = f(2) = a$, $f(3) = b$

Show that f is not continuous, not open, closed.

Example (8) :

Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{a\}\}$

Define the function $f : (X, \tau) \to (Y, \tau')$; $f(1) = f(2) = a$, $f(3) = b$

Show that f is not continuous, not open, not closed.